Question

If a particle is projected vertically upwards to a height h above a point on the ground at a northern latitude λ, show that it strikes the ground at a point

$$\frac{4}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}}$$

to the west where ω is the angular velocity of the earth (Neglect air resistance and consider only small vertical heights).

Answer

Usual coordinate system - see question 2.
Newton's 2nd law: $m\ddot{\mathbf{r}} = m\mathbf{g} - 2m\boldsymbol{\omega}\times\mathbf{v}$
$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \qquad \boldsymbol{\omega} = \omega(-\cos\lambda\mathbf{i} + \sin\lambda\mathbf{k})$

$$\begin{aligned}
\boldsymbol{\omega}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} \\
&= -\omega\sin\lambda\dot{y}\,\mathbf{i} + \omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda)\mathbf{j} + (-\omega\dot{y}\cos\lambda)\mathbf{k}
\end{aligned}$$

$$g = -g\mathbf{k}$$

Therefore putting all this into Newton's 2nd law gives:
$m(\ddot{x}\mathbf{i}+\ddot{y}\mathbf{j}+\ddot{z}\mathbf{k}) = -mg\mathbf{k} - 2m\omega[-\sin\lambda\dot{y}\mathbf{i} + (\dot{x}\sin\lambda + \dot{z}\cos\lambda)\mathbf{j} + (-\dot{y}\cos\lambda)\mathbf{k}]$
Equating components gives:

$$\begin{aligned}
\ddot{x} &= 2\omega\dot{y}\sin\lambda \\
\ddot{y} &= -2\omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda) \\
\ddot{z} &= 2\omega\dot{y}\cos\lambda - g
\end{aligned}$$

First approximation: put $\omega = 0 \Rightarrow \ddot{x} = \ddot{y} = 0 \quad \ddot{z} = -g$
Therefore $x = y = 0 \quad z = Ut - \frac{1}{2}gt^2$ where U is the initial upward speed.
The particle reaches height h when $\dot{z} = 0$ therefore $t = \frac{U}{g}$. Hence

$$h = U\frac{U}{g} - \frac{1}{2}g\frac{U^2}{g^2} = \frac{U^2}{2g}$$

The time of flight is $\quad t = \dfrac{2U}{g}$.
Next approximation: (insert first approximation in Coriolis terms).

$$\begin{aligned}
\ddot{x} &= 0 \\
\ddot{y} &= -2\omega\cos\lambda(U - gt) \\
\ddot{z} &= -g
\end{aligned}$$

Thus

$$
\begin{aligned}
y &= 2\omega\cos\lambda\left(\frac{1}{6}gt^3 - \frac{1}{2}Ut^2\right) \\
\text{when} \quad t &= 2\frac{U}{g} \quad \text{the particle returns to the ground} \\
&= 2\omega\cos\lambda\frac{1}{6}\frac{4U^2}{g^2}\left(g\cdot\frac{2U}{g} - 3U\right) \\
&= -\frac{4}{3}\omega\cos\lambda\frac{U^3}{g^2} \\
&= -\frac{4}{3}\omega\cos\lambda\frac{(2gh)^{\frac{3}{2}}}{g^2} \\
&= -\frac{4}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}} \qquad \text{(to the west as negative)}
\end{aligned}
$$