QUESTION

(a) Evaluate the integral

$\int_\gamma z^n \, dz$

where n is any integer and γ is the circle of radius r about the origin. (Note that you will have to treat the two cases $n = -1$ and $n \neq 1$ separately.)

(b) What does this tell you about the residue theorem?

ANSWER

(a) $z(t) = re^{it}, \quad 0 \leq t \leq 2\pi, \quad \frac{dz}{dt} = ire^{it}$

$$\begin{aligned}
\int_\gamma z^n \, dz &= \int_0^{2\pi} (re^{it})^r ire^{it} \, dt \\
&= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt \\
&= \begin{cases} ir^{n+1} \left[\frac{e^{i(n+1)t}}{i(n+1)} \right]_0^{2\pi} = 0 & \text{for } n \neq -1 \\ i \, [t]_0^{2\pi} = 2\pi i & \text{for } n = -1 \end{cases}
\end{aligned}$$

(b) If we write $f(z) = \sum_{n=-\infty}^{\infty} c_{-n} z^n$ for $|z| \leq r$

then $\int_\gamma f(z) \, dz = \sum_{n=-\infty}^{\infty} C_{-n} \int_\gamma z^n \, dz = 2\pi i c_{-1}$