Question

Find the Cartesian and vector equations of the plane passing through the points (1,1,1),(1,2,3) and (3,2,1).

Answer

Let

$$\begin{aligned}\mathbf{OA} &= (1,1,1) = \mathbf{i}+\mathbf{j}+\mathbf{k}\\ \mathbf{OB} &= (1,2,3) = \mathbf{i}+2\mathbf{j}+3\mathbf{k}\\ \mathbf{OC} &= (3,2,1) = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}\end{aligned}$$

Need two vectors in the plane, say $\mathbf{AB},\ \mathbf{BC}$

$\mathbf{AB} = \mathbf{AO}+\mathbf{OB} = -\mathbf{i}-\mathbf{j}-\mathbf{k}+\mathbf{i}+2\mathbf{j}+3\mathbf{k} = \mathbf{j}+2\mathbf{k}$

$\mathbf{BC} = \mathbf{BO}+\mathbf{OC} = -\mathbf{i}-2\mathbf{j}-3\mathbf{k}+3\mathbf{i}+2\mathbf{j}+\mathbf{k} = 2\mathbf{i}-2\mathbf{k}$

A vector to a point in the plane is $\mathbf{OA} = \mathbf{i}+\mathbf{j}+\mathbf{k}$

Thus a vector equation is

$$\begin{aligned}\mathbf{r} &= \mathbf{OA}+\lambda(\mathbf{AB})+\mu(\mathbf{BC})\\ \Rightarrow &= \underline{\mathbf{i}+\mathbf{j}+\mathbf{k}+\lambda(\mathbf{j}+2\mathbf{k})+\mu(2\mathbf{i}-2\mathbf{k})}\end{aligned}$$

We now need $\hat{\mathbf{n}}$ a vector normal to plane ABC

$$\hat{\mathbf{n}} = \frac{\mathbf{AB}\times\mathbf{BC}}{|\mathbf{AB}\times\mathbf{BC}|}$$

$$\begin{aligned}&\mathbf{AB}\times\mathbf{BC}\\ =& \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 2\\ 2 & 0 & -2\end{vmatrix}\\ =& \mathbf{i}(1\times -2)-\mathbf{i}(0\times 2)-\mathbf{j}(0\times -2)\\ &+\mathbf{k}(0\times 0)+\mathbf{j}(2\times 2)-\mathbf{k}(2\times 1)\\ =& -2\mathbf{i}+4\mathbf{j}-2\mathbf{k}\end{aligned}$$

$|\mathbf{AB}\times\mathbf{BC}| = \sqrt{(-2)^2+4^2+(-2)^2} = \sqrt{4+16+4} = \sqrt{24}$

so $\hat{\mathbf{n}} = -\frac{2}{\sqrt{24}}\mathbf{i}+\frac{4}{\sqrt{24}}\mathbf{j}-\frac{2}{\sqrt{24}}\mathbf{k} = -\frac{\mathbf{i}}{\sqrt{6}}+2\frac{\mathbf{j}}{\sqrt{6}}-\frac{\mathbf{k}}{\sqrt{6}}$

The perpendicular distance d from 0 to the plane is $d = \mathbf{r}\cdot\hat{\mathbf{n}}$ where $\mathbf{r}$ is the position vector of any point in the plane, say $\mathbf{OA} = \mathbf{i}+\mathbf{j}+\mathbf{k}$. Thus

$$d = (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\left(-\frac{\mathbf{i}}{\sqrt{6}}+2\frac{\mathbf{j}}{\sqrt{6}}-\frac{\mathbf{k}}{\sqrt{6}}\right) = -\frac{1}{\sqrt{6}}+2\frac{2}{\sqrt{6}}-\frac{1}{\sqrt{6}} = \underline{0}$$

Thus the plane actually passes through the origin!
Thus we have for a general point $(x, y, z) = \mathbf{r}$,
$(x, y, z) \cdot \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = 0$
$\Rightarrow -x + 2y - z = 0$
or$\underline{\mathrm{x - 2y + x = 0}}$