Question
A man with $n$ keys wants to open his door and tries the keys at random. Exactly one key will open the door. Let $X$ denote the number of trials required to open the door for the first time. Find $E(X)$ if

(a) unsuccessful keys are not eliminated from further selections

(b) unsuccessful keys are eliminated

Answer
Let $X$ be the number of trials needed to open the door. Let ‘$S$’ denote success i.e. the door is opened and ‘$F$’ denote failure for each trial.

(a) The event $X = x$ is equivalent to the event $F F F \ldots F S$

Also $P(S) = \frac{1}{n}$ and $P(F) = \frac{n-1}{n}$.

$X$ has the geometric distribution with $p = P(S) = \frac{1}{n}$.

Therefore $E(X) = \frac{1}{p} = \frac{1}{\frac{1}{n}} = n$

(b) If unsuccessful keys are eliminated then it can take at most $n$ attempts to open the door with the following probabilities:

\[
\begin{align*}
P(X = 1) &= \frac{1}{n} \\
P(X = 2) &= \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n} \\
P(X = 3) &= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{1}{n-2} = \frac{1}{n} \\
&\vdots \\
P(X = n) &= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{n}.
\end{align*}
\]

Therefore $E(X) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + \ldots + n \cdot \frac{1}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$. 