**Question**

Evaluate the following integrals using the method of residues.

i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \qquad -1 < a < 1$

ii) $\displaystyle\int_0^{\infty} \frac{\cos bx dx}{(1+x^2)^2} \qquad b > 0.$

In (ii) describe briefly any convergence properties that you use.

**Answer**

i) Let $z = e^{i\theta}, \qquad dz = iz d\theta$

$\cos\theta = \frac{1}{2}(z + \frac{1}{z}) \qquad C$ - the unit circle.

$$I = \int_0^2 \pi \frac{d\theta}{1 + a\cos\theta} = \int_C \frac{dz}{iz\left(1 + \frac{a}{z}\left(z + \frac{1}{z}\right)\right)}$$

$$= \frac{1}{i}\int_C \frac{dz}{z + \frac{a}{2}z^2 + \frac{a}{2}} = \frac{2}{ia}\int_C \frac{dz}{z^2 + \frac{2}{a}z + 1}$$

The denominator has roots $-\dfrac{1}{a} \pm \dfrac{1}{a}\sqrt{1-a^2}$, so the integrand has a simple pole inside $C$ at $z = -\dfrac{1}{a} + \dfrac{1}{a}\sqrt{1-a^2}$, with residue

$$\left.\frac{1}{z - \left(-\frac{1}{a} - \frac{1}{a}\sqrt{1-a^2}\right)}\right|_{z=-\frac{1}{a}+\frac{1}{a}\sqrt{1-a^2}} = \frac{1}{\frac{2}{a}\sqrt{1-a^2}} = \frac{a}{2\sqrt{1-a^2}}$$

So $I = 2\pi i \dfrac{2}{ia}\dfrac{a}{2\sqrt{1-a^2}} = \dfrac{2\pi}{\sqrt{1-a^2}}$

ii) $I = \displaystyle\int_0^{\infty} \frac{\cos bx}{(1+x^2)^2} dx \qquad \left|\frac{\cos bx}{(1+x^2)^2}\right| \le \frac{1}{x^4}$ and $\displaystyle\int^{\infty} \frac{1}{x^4}$ converges.

and $2I = \displaystyle\int_{-\infty}^{\infty} \frac{\cos bx}{(1+x^2)^2}.$

Let $f(z) = \dfrac{e^{ibz}}{(1+z^2)^2}$, then $f(z)$ has poles of order 2 at $z = \pm i$.

We integrate $f(z)$ round $\Gamma$

DIAGRAM

$$\mathrm{res}\, i = \frac{d}{dz}(z-i)^2 f(z)\bigg|_{z=i} = \frac{d}{dz}\frac{e^{ibz}}{(z+i)^2}\bigg|_{z=i} = \frac{-i(b+1)e^{-b}}{4}$$

On $C$, $z = Re^{it}$, and $|e^{ibz}| = |e^{-bR\sin t}| \leq 1$ and $|f(z)| \leq \dfrac{1}{(R^2-1)^2}$

so $\left|\int_C f(z)dz\right| \leq \dfrac{\pi R}{(R^2-1)^2} \to 0$ as $R \to \infty$

So $\int_\Gamma f(z)dz = 2\pi i \dfrac{-i(b+1)e^{-b}}{4} = \dfrac{\pi(b+1)e^{-b}}{2}$

letting $R \to \infty$ then gives $I = \dfrac{\pi(b+1)e^{-b}}{4}$