Question

Derive the Cauchy-Riemann equations as necessary conditions for the function

$$f(z) = u(x, y) + iv(x, y), \quad \text{where} \quad z = x + iy$$

to be analytic. State without proof sufficient conditions for the differentiability of f in terms of the partial derivatives of u and v.

Let $f(z) = \dfrac{z}{\bar{z}}, \qquad (z \neq 0)$

Show that f is differentiable nowhere in $\mathbf{C} - \{0\}$. Is it possible to extend the definition of f to 0 in such a way that f is differentiable at 0? Justify your answer.

Answer

Derivation of Cauchy-Riemann equations is bookwork.

Let $f(z) = \dfrac{z}{\bar{z}} = \dfrac{x + iy}{x - iy} = (x + iy)^2 x^2 + y^2 = \dfrac{x^2 - y^2}{x^2 + y^2} + i\dfrac{2xy}{x^2 + y^2} \qquad z \neq 0$

$\dfrac{\partial u}{\partial x} = \dfrac{(x^2 + y^2)2x - (x^2 - y^2)2x}{(x^2 + y^2)^2} = \dfrac{4xy^2}{(x^2 + y^2)^2}$

$\dfrac{\partial v}{\partial y} = \dfrac{(x^2 + y^2)2y - 2xy2x}{(x^2 + y^2)^2} = \dfrac{2y^3}{(x^2 + y^2)^2}$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ iff $4xy^2 = 2y^3$ i.e. $4x = 2y$ or $y = 0$

$\dfrac{\partial u}{\partial y} = \dfrac{(x^2 + y^2)(-2y) - (x^2 - y^2)2y}{(x^2 + y^2)^2} = \dfrac{-4x^2 y}{(x^2 + y^2)^2}$

$\dfrac{\partial v}{\partial x} = \dfrac{(x^2 + y^2)2x - 2xy2y}{(x^2 + y^2)^2} = \dfrac{2x^3}{(x^2 + y^2)^2}$

$\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ iff $4x^2 y = 2x^3$ i.e. $4y = 2x$ or $x = 0$

The conditions cannot be satisfied for $z \neq 0$.

Now in polars $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$. So $\dfrac{z}{\bar{z}} = e^{2i\theta}$

For $\theta = 0$, $\dfrac{z}{\bar{z}} = 1$ and for $\theta = \dfrac{\pi}{2}$, $\dfrac{z}{\bar{z}} = -1$

So $f(z)$ does not have a limit as $z \to 0$ and so cannot be defined to be differentiable or continuous at $z = 0$.