Question

(a) Let $z = x + iy$, $\log z$ be defined with $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$, and set C to be the positively oriented contour shown in the diagram below.

PICTURE

Show that

$$J = \oint_C dz\ \frac{\log(z+i)}{z^2+1} = \pi \log 2 + i\frac{\pi^2}{2}.$$

You should carefully show in the diagram where any branch cuts in the integrand are located.

(b) By considering J and the relationship between the contributions from $-R \le x \le 0$ and $0 \le x \le R$, and other factors show that

$$\int_0^\infty dx\ \frac{\log(x^2+1)}{x^2+1} = \pi \log 2.$$

You may assume that $R\frac{|\log(Re^{i\theta}+i)|}{|R^2-1|} \to 0$ as $R \to \infty$,

$0 \le \theta \le \pi$.

Answer

$$I = \oint_C dz\ \frac{\log(z+i)}{z^2+1}$$

(a) $\oint_C = 2\pi i \times residue \left(\frac{\log(z+i)}{z^2+1}\right)$ at $z = +i$

PICTURE

$$2\pi i \lim_{z\to i}\left[\frac{\log(z+i)}{(z+i)(z-i)}\times(z-i)\right]$$

$$\begin{aligned}
&= 2\pi i\frac{\log(2i)}{2i}\\
&= \pi[\log|2i| + i\arg(2i)]\\
&= \pi\log 2 + i\pi\times\frac{\pi}{2}\\
&\quad \text{since } -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\\
&= \underline{\pi\log 2 + \frac{i\pi^2}{2}}
\end{aligned}$$

(b) $\oint_C = \int_0^R + \int_{z=|R|\ Im(z)>0} + \int_{-R}^0$

Now $\int_0^R = \int_0^R \frac{dx\ \log(x+i)}{(x^2+1)}$

and $\int_{-R}^0 = -\int_R^0 \frac{dx\ \log(-x+i)}{(-x)^2+1} = \int_0^R \frac{dx\ \log(i-x)}{x^2+1}$

$z=-x$

$$\begin{aligned}
\int_{z=|R|\ Im(z)>0} &= i\int_{\theta=0}\pi d\theta\ \frac{e^{i\theta}R\log(Re^{i\theta}+i)}{(R^2e^{2i\theta}+1)}\\
&\leq \left|\int_0^\pi d\theta\ \frac{R\log(Re^{i\theta}+i)}{(R^2e^{2i\theta}+1)}\right|\\
&\leq \int_0^\pi d\theta\ R\ \frac{|\log(Re^{i\theta}+i)}{|R^2e^{2i\theta}+1|}\\
&\leq \int_0^\theta d\theta\ \frac{R|\log(Re^{i\theta}+i)|}{|R^2-1|}\\
&\quad \text{since } |\mathrm{R}^2\mathrm{e}^{2\mathrm{i}\theta}+1| \geq ||\mathrm{R}^2\mathrm{e}^{2\mathrm{i}\theta}|-|1|| = |\mathrm{R}^2-1|\\
&\leq \frac{\pi R|\log(Re^{i\theta}+i)|}{|R^2-1|}\\
&\to 0 \text{ as } \mathrm{R}\to\infty \text{ by hint}
\end{aligned}$$

Therefore $J = \lim_{R\to\infty}\int_0^R dx\ \frac{\log(x+i)}{x^2+1} + \lim_{R\to\infty}\int_0^R dx\ \frac{\log(i-x)}{x^2+1}$

Therefore

$$
\begin{aligned}
J &= \int_0^\infty \frac{dx}{(x^2+1)}[\log(x+i)+\log(i-x)] \\
&= \int_0^\infty \frac{dx}{x^2+1}\log(ix-x^2-1-ix) \\
&= \int_0^\infty \frac{dx}{(x^2+1)}\log(-x^2-1) \\
&= \int_0^\infty \frac{dx}{(x^2+1)}\log[(1+x^2)\times -1] \\
&= \int_0^\infty \frac{dx}{(x^2+1)}\log(1+x^2) + i\pi\int_0^\infty \frac{dx}{(x^2+1)}
\end{aligned}
$$

So

$$
\underline{\int_0^\infty \frac{dx}{(x^2+1)}\log(1+x^2) = Re(J) = \pi\log 2}
$$