Question

Solve the following differential equations subject to the given initial conditions:

1. $y'' + y' - 2y = 2e^x \quad y(0) = 0 \quad y'(0) = 1$
2. $y'' - 2y' + y = e^x + 4 \quad y(0) = 1 \quad y'(0) = 1 \qquad (*)$
3. $y'' + 2y' + 5y = 4e^{-x} \cos 2x \quad y(0) = 1 \quad y'(0) = 0$

Answer

1. Auxiliary equation is: $m^2 + m - 2 = 0 \qquad$ giving $m = -2, 1$
   Hence the complementary function is: $y_{cf} = Ae^{-2x} + Be^x$
   Note that the right hand side of the equation contains a linear combination of the complementary function so try the particular integral: $y_p = axe^{2x}$
   Equation becomes: $(2ae^x + axe^x) + (ae^x + axe^x) - 2\,(axe^x) = 2e^x$
   Hence $3ae^x = 2e^x \qquad$ so that $\qquad$ a=2/3
   The general solution is: $y = Ae^{-2x} + Be^x + \dfrac{2}{3}xe^x$
   Imposing $y(0) = 1$ gives $\qquad A + B = 0$
   Imposing =$y'(0) = 1$ gives $\qquad -2A + B + 2/3 = 1$
   Hence $A = \dfrac{-1}{9}$ and $B = \dfrac{1}{9}$ Hence $y = \dfrac{1}{9}\left(e^x - e^{-2x}\right) + \dfrac{2}{3}xe^x$

2. Auxiliary equation is: $m^2 - 2m + 1 = 0 \qquad$ giving $m = 1, 1$
   Hence the complementary function is: $y_{cf} = e^x(A + Bx)$
   Note that the right hand side contains a term, e^x which is in the complementary function and that xe^x is also on the complementary function. Hence try the particular integral: $y_p = ax^2e^x + b$
   (the a term allows for the e^x in the right hand side and the b term allows for the constant, 4.) Equation becomes:
   $\left(2ae^x + 4axe^x + ax^2e^x\right) - 2\left(2axe^x + ax^2e^x\right) + \left(ax^2e^x + b\right) = e^x + 4$
   and simplifying gives: $2ae^{-x} + b = e^{-x} + 4$ so that $a = 1/2$ and $b = 4$
   The general solution is: $y = e^x(A + Bx) + \dfrac{1}{2}x^2e^x + 4$
   Imposing $y(0) = 1$ gives $\qquad A + 4 = 1$

Imposing $y'(0) = 1$ gives $\quad A + B = 1$
so that $A = -3$ and $B = 4$.
Giving the solution $y = e^x(4x - 3) + \dfrac{1}{2}x^2 e^x + 4$

3. Auxiliary equation is: $m^2 + 2m + 5 = 0 \quad$ giving $m = -1 \pm 2i$
Hence the complementary function is: $y_{cf} = e^{-x}(A\cos(2x) + B\sin(2x))$
Note that the right hand side of the equation contains a linear combination of the complementary function so:
try the particular integral: $y_p = axe^{-x}\sin(2x) + bxe^{-x}\cos(2x)$
Equation becomes:
$\Big(a(-2e^{-x}\sin(2x) + 4e^{-x}\cos(2x) - 4xe^{-x}\cos(2x) - 3xe^{-x}\sin(2x))$
$+b(-4e^{-x}\sin(2x) - 2e^{-x}\cos(2x) - 3xe^{-x}\cos(2x) + 4xe^{-x}\sin(2x))\Big)$
$+2\Big(a(e^{-x}\sin(2x) + 2xe^{-x}\cos(2x) - xe^{-x}\sin(2x)) + b(e^{-x}\cos(2x) - 2xe^{-x}\sin(2x) -$
$xe^{-x}\cos(2x))\Big) + 5\Big(axe^{-x}\sin(2x) + bxe^{-x}\cos(2x)\Big) = 4e^{-x}\cos(2x)$
Hence
$\Big(a(-2e^{-x}\sin(2x) + 4e^{-x}\cos(2x)) + b(-4e^{-x}\sin(2x) - 2e^{-x}\cos(2x))\Big)$
$+ 2\Big(a(e^{-x}\sin(2x)) + b(e^{-x}\cos(2x))\Big) = 4e^{-x}\cos(2x)$
so that $\quad a = 1$ and $b = 0$
The general solution is: $y = e^{-x}(A\cos(2x) + B\sin(2x)) + xe^{-x}\sin 2x$
Imposing $y(0) = 1$ gives $\quad A = 1$
Imposing $y'(0) = 0$ gives $\quad -A + 2B = 0$ or $B = 1/2$
Hence $y = e^{-x}(\cos(2x) + \dfrac{1}{2}\sin(2x)) + xe^{-x}\sin 2x$

(note that the particular integral for this last question can also be found by writing the problem in the form

$$y'' + 2y' + 5y = 2\left(e^{(-1+2i)x} + e^{(-1-2i)x}\right)$$

and then seeking a particular integral of the form $y_p = axe^{(-1+2i)x} + bxe^{(-1-2i)x}$
but care must be taken as both a and b here will be complex and to get a real solution we shall require $b = a^*$ (where * means complex conjugate).)