Question
Solve the following initial value problems:

1. $9y'' - 12y' + 4y = 0 \quad y(0) = 2 \quad y'(0) = -1$
2. $y'' + 4y' + 3y = 0 \quad y(0) = 2 \quad y'(0) = -1$
3. $y'' + 4y' + 5y = 0 \quad y(0) = 1 \quad y'(0) = 0 \qquad (*)$
4. $y'' + 4y' + 4y = 0 \quad y(-1) = 2 \quad y'(-1) = 1 \qquad (*)$

Answer

1. We guess that the solution has the form $y = e^{mx}$ so that the auxiliary equation is: $9m^2 - 12m + 4 = 0$
   which has roots $m = \dfrac{12 \pm \sqrt{144 - 144}}{18} = \dfrac{2}{3}, \dfrac{2}{3}$
   As the root is repeated the general solution is:
   $y = Ae^{2x/3} + Bxe^{2x/3} = (A + Bx)e^{2x/3}$
   Let $y(0) = 2$ so that $A = 2$
   Let $y'(0) = -1$ so that using $y' = 2A/3e^{2x/3} + 2Bx/3e^{2x/3} + Be^{2x/3}$
   we require $2A/3 + B = -1$
   Hence $A = 2$, $B = -7/3$ and the solution is:
   $$(2 - 7x/3)e^{2x/3}$$
2. Auxiliary equation is: $m^2 + 4m + 3 = 0$
   which has roots $m = \dfrac{-4 \pm \sqrt{16 - 12}}{2} = -1, -3$
   The general solution is: $y = Ae^{-x} + Bxe^{-3x}$
   Let $y(0) = 2$ so that $A + B = 2$
   Let $y'(0) = -1$ so that $-A - 3B = -1$
   Hence $A = 5/2$, $B = -1/2$ and the solution is:
   $$y = \frac{5}{2}e^{-x} - \frac{1}{2}e^{-3x}$$
3. Auxiliary equation is: $m^2 + 4m + 5 = 0$
   which has roots $m = \dfrac{-4 \pm \sqrt{16 - 20}}{2} = -2 + i, -2 - i$

The general solution is: $y = e^{-2x}(A\cos x + B\sin x)$
Let $y(0) = 1$ so that $A = 1$
Let $y'(0) = 0$ so that $-2A + B = 0$
Hence $A = 1$, $B = 2$ and the solution is:

$$y = e^{-2x}(\cos x + 2\sin x)$$

4. Auxiliary equation is: $m^2 + 4m + 4 = 0$
   which has roots $m = \dfrac{-4 \pm \sqrt{16-16}}{2} = -2, -2$
   As the roots are repeated the general solution is:
   $y = e^{-2x}(A + Bx)$
   Let $y(-1) = 2$ so that $e^2(A - B) = 2$
   Let $y'(-1) = 1$ so that $-2e^2(A - B) + e^2 B = 1$
   Hence $A = 7e^{-2}$, $B = 5e^{-2}$ and the solution is:

$$y = e^{-2x}(7e^{-2} + 5xe^{-2}) = (5x + 7)e^{(-2(x+1))}$$