QUESTION

**(i)** Prove that a positive integer $n$ is composite if and only if it is divisible by some prime $p$ such that $p \leq \sqrt{n}$.

**(ii)** Design a test for deciding when $n$ is the product of at most two primes, including the possibility that $n = p^2$ for some prime $p$.

**(iii)** Use your test from (ii) to find all the integers $n$ in the range $600 \leq n \leq 620$ which are either prime or the product of two (not necessarily distinct) primes.

**(iv)** Use (i) together with your result in (iii) to fine all the primes $p$ in the range $600 \leq p \leq 620$.

ANSWER

**(i)** If $n$ is composite then $n = p_1 p_2 \ldots p_s$ where the $p_i$ are (not necessarily distinct) primes. If $p_1$ is the smallest $p_i$ then $p_1^2 \leq p_1 p_2 \leq n$ so that $p_1 \leq \sqrt{n}$. Conversely, a prime divisor in the range $2 \leq p \leq \sqrt{n}$ must be a proper divisor.

**(ii)** If $n$ is composite then $n = p_1 p_2 \ldots p_s$ where the $p_i$ are (not necessarily distinct) primes and if $s \geq 3$ we must have $p_1^3 \leq p_1 p_2 p_3 \leq n$, where $p_1$ is the smallest $p_i$. Therefore if one attempts unsuccessfully to divide $n$ by each of the primes in the range $2 \leq p \leq n^{\frac{1}{3}}$ then $n$ is the product of at most two primes.

**(iii)** Since $729 = 9^3$ we need only eliminate from the set $600 \leq n \leq 620$ all multiple of 2,3,5 and 7. Deleting all multiple of 2,3,5 leaves

$$611, 613, 617, 619$$

and none of these are divisible by 7 since $600 \equiv 5$ (modulo 7) and none of 16,18,22,24 are divisible by 7.

**(iv)** Since $25^2 = 625$ we need only test the divisibility of 611,613,617, 619 by the primes 11,13,17,19,23 (having already dealt with 2,3,5 and 7). None are divisible by 11. However $13 \times 47 = 611$ so that 613, 617, 619 are not divisible by 13. None of 613,617,619 is divisible by 17,19 or 23. Therefore 613,617 and 619 are all prime.