Question

Let $A \subseteq \mathbf{R^n}$ and $B \subseteq \mathbf{R^m}$. Let m_n^* denote Lebesgue outer measure in $\mathbf{R^n}$. Show that $m_{n+m}^*(A \times B) = m_n^*(A) \cdot m_n^*(B)$ where $A \times B$ is the cartesian product.

Answer

Let $\bigcup R_i \supseteq A$ and $\bigcup S_j \supseteq B$.
Suppose $m^*(A) < \infty$ and $m^*(B) < \infty$
Then $\bigcup\limits_{ij} R_i \times S_j \supseteq A \times B$
Choose $\{R_i\}$ so that $\sum |R_i| \leq m^*(A) + \epsilon$
Choose $\{S_j\}$ so that $\sum |S_j| \leq m^*(B) + \epsilon$
Then $\sum |R_i \times S_j| = \sum |R_i||S_j| = \sum |R_i| \sum |S_j|$
$\leq m^*(A)m^*(B) + \epsilon_1$
Therefore $m^*(A \times B) \leq m^*(A)m^*(B) + \epsilon$
Now cover $(A \times B)$ by $R_i \subseteq \mathbf{R^{m+n}}$
Let S_i be the projection of R_i onto $\mathbf{R^n}$
Let T_i be the projection of R_i onto $\mathbf{R^m}$
Then $\bigcup S_i \supseteq A$, and $\bigcup T_i \supseteq B$
Choose R_i so that $\sum |R_i| \leq m^*(A \times B) + \epsilon$
$m^*(A)m^*(B) \leq \sum |S_i| \sum |T_j| = \sum |R_i| \leq m^*(A \times B) + \epsilon$
Hence the result. Deal with infinite measure cases using σ-finiteness arguments.