Question

The three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent.

(a) What does the equation

$$\mathbf{r} = (1 + 2u - v)\mathbf{a} + (3v - 6u)\mathbf{b} + \mathbf{c}$$

$u, v \epsilon \mathbf{R}$ represent?

(b) When $\mathbf{p} = \mathbf{a} + 2\mathbf{c}$, $\mathbf{q} = 2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$, $\mathbf{s} = 2\mathbf{b} - 3\mathbf{a} - 4\mathbf{c}$ are $\mathbf{p}, \mathbf{q}, \mathbf{s}$ linearly independent? If $\mathbf{p}, \mathbf{q}, \mathbf{s}$ are position vectors of the points P, Q, S are P, Q, S colinear?

(c) Find the point(s) of intersection of the line

$$\mathbf{r} = (1 + t)\mathbf{a} + (t - 2)\mathbf{b} + (2 - t)\mathbf{c} \qquad t \epsilon \mathbf{R}$$

with

(i) The lines

$$\begin{array}{llll} (a) & \mathbf{r} = & (2 + u)\mathbf{a} + (u - 1)\mathbf{b} + (2u + 1)\mathbf{c} & u \epsilon \mathbf{R} \\ (b) & \mathbf{r} = & (1 + u)(2\mathbf{a} + \mathbf{b}) + (3 + u)\mathbf{c} & u \epsilon \mathbf{R} \\ (c) & \mathbf{r} = & u\mathbf{a} + (u - 3)\mathbf{b} + (3 - u)\mathbf{c} & u \epsilon \mathbf{R} \end{array}$$

(ii) The planes

$$\begin{array}{llll} (a) & \mathbf{r} = & \mathbf{a} + u\mathbf{b} + v\mathbf{c} & u, v \epsilon \mathbf{R} \\ (b) & \mathbf{r} = & (1 + u)\mathbf{a} + (u - 2)\mathbf{b} + v\mathbf{c} & u, v \epsilon \mathbf{R} \\ (c) & \mathbf{r} = & (1 + v)\mathbf{a} + (u - 2)\mathbf{b} + (3 - v)\mathbf{c} & u, v \epsilon \mathbf{R} \end{array}$$

Answer

(a)

$$\begin{aligned} bfr &= (1 + 2u - v)\mathbf{a} + (3v - 6v)\mathbf{b} + \mathbf{c} \\ &= \mathbf{a} + \mathbf{c} + u(2\mathbf{a} - 6\mathbf{b}) + v(3\mathbf{b} - \mathbf{a}) \\ &= \mathbf{a} + \mathbf{c} + (2u - v)(\mathbf{a} - 3\mathbf{b}) \end{aligned}$$

This is a line through $\mathbf{a} + \mathbf{c}$ with direction vector $\mathbf{a} - 3\mathbf{b}$

(b)

$$\mathbf{p} = \mathbf{a} + 2\mathbf{c} \qquad \mathbf{q} = 2\mathbf{a} - \mathbf{b} + 3\mathbf{c} \qquad \mathbf{s} = 2\mathbf{b} - 3\mathbf{a} - 4\mathbf{c}$$

$$\alpha\mathbf{p} + \beta\mathbf{q} + \gamma\mathbf{s} = \mathbf{a}(\alpha + 2\beta - 3\gamma) + \mathbf{b}(-\beta + 2\gamma) + \mathbf{c}(2\alpha + 3\beta - 4\gamma) = \mathbf{0}$$

if and only if

$$\left.\begin{array}{rcl} \alpha + 2\beta - 3\gamma & = & 0 \\ -\beta + 2\gamma & = & 0 \\ 2\alpha + 3\beta - 4\gamma & = & 0 \end{array}\right\} \text{Solution } (-\gamma, 2\gamma, \gamma)$$

So $\mathbf{p}$, $\mathbf{q}$, $\mathbf{s}$, are not linearly independent.

$\alpha + \beta + \gamma = 2\gamma = 0$ iff $\alpha = \beta = \gamma = 0$. So PQR are not collinear.

(c) $\mathbf{r} = (1+t)\mathbf{a} + (t-2)\mathbf{b} + (2-t)\mathbf{c}$

(in standard form $\mathbf{r} = \mathbf{a} - 2\mathbf{b} + 2\mathbf{c} + t(\mathbf{a} + \mathbf{b} - \mathbf{c})$)

(i) **(a)** $\mathbf{r} = (1+t)\mathbf{a} + (t-2)\mathbf{b} + (2-t)\mathbf{c}$

$\mathbf{r} = (2+u)\mathbf{a} + (u-1)\mathbf{b} + (2u+1)\mathbf{c}$

These lines meet where

$$\begin{array}{rclrclrcl} 1+t & = & 2+u & t-2 & = & u-1 & 2-t & = & 2u+1 \\ t & = & 1+u & t & = & 1+u & t & = & 1-2u \end{array}$$

So we require $1 + u = 1 + 2u$ $v = 0$ so $t = 1$

Thus the lines meet at $\mathbf{r} = 2\mathbf{a} - \mathbf{b} + \mathbf{c}$

(b) $\mathbf{r} = (1+t)\mathbf{a} + (t-2)\mathbf{b} + (2-t)\mathbf{c}$

$\mathbf{r} = (1+u)(2\mathbf{a} + \mathbf{b}) + (3+u)\mathbf{c}$

These meet where

$$(1+t) = (2+2u) \qquad \underbrace{(t-2) = (1+u) \qquad (2-t) = 3+u}_{1+u = -3-3u \Rightarrow u = -2,\ t = 1}$$

which doesn't fit 1st equation.

Therefore the lines do not meet.

(c) $\mathbf{r} = (1+t)\mathbf{a} + (t-2)\mathbf{b} + (2-t)\mathbf{c}$

$\mathbf{r} = u\mathbf{a} + (u-3)\mathbf{b} + (3-u)\mathbf{c}$

$u = 1 + t$ so the lines are identical.

(ii) **(a)** $\mathbf{r} = (1+t)\mathbf{a} + (t-2)\mathbf{b} + (2-t)\mathbf{c}$

$\mathbf{r} = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$

$$\begin{array}{rclrclrcl} 1+t & = & 1, & t-2 & = & u, & 2-t & = & v \\ t & = & 0, u & & = & -2, & v & = & 2 \end{array}$$

So $\mathbf{r} = \mathbf{a} - 2\mathbf{2} + 2\mathbf{c}$ is the point of intersection.

(b) $\mathbf{r} = (1+t)\mathbf{a} + (t-2)\mathbf{b} + (2-t)\mathbf{c}$

$\mathbf{r} = (1+u)\mathbf{a} + (u-2)\mathbf{b} + v\mathbf{c}$

$t = u$, $v = -t + 2$ So the line lies in the plane.

(c) $\mathbf{r} = (1+t)\mathbf{a} + (t-2)\mathbf{b} + (2-t)\mathbf{c}$

$\mathbf{r} = (1+v)\mathbf{a} + (u-2)\mathbf{b} + (3-v)\mathbf{c}$

$t = v$, $t - u\,t = v - 1$

So the line does not meet the plane. It is parallel to the plane.