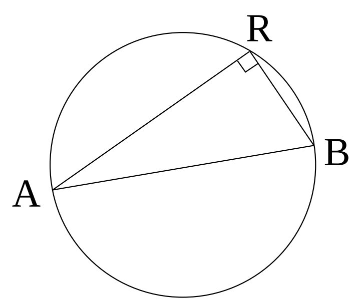
Question
Using vector methods

(a) find the equation of a circle on AB as diameter,

(b) prove that the altitudes of a triangle are concurrent,

(c) show that the diagonals of a rhombus are orthogonal.

Answer



(a)

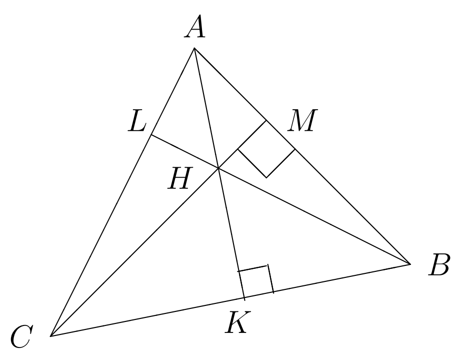
A, B, R have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{r}$ R lies on the circle.

Diameter AB iff RA is perpendicular to RB.

$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$

$\mathbf{r} \cdot \mathbf{r} - \mathbf{r} \cdot (\mathbf{a} + \mathbf{b}) + (\mathbf{a} \cdot \mathbf{b}) = 0$

(b)



CM is perpendicular to AB.

AK is perpendicular to BC

H is $AK \cap CM$

L if $BH \cap AC$

prove BL is perpendicular to AC

Let H be the origin and let the position vectors of ABC be $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$.

$$\begin{aligned} \text{Then } M &= m\mathbf{c} \quad \text{for some} \quad m \neq 0 \\ K &= k\mathbf{a} \qquad\qquad\quad k \neq 0 \\ L &= l\mathbf{b} \qquad\qquad\quad\, l \neq 0 \end{aligned}$$

Since HM is perpendicular to AB so $m\mathbf{c} \cdot (\mathbf{b} - \mathbf{a}) = 0$

therefore $\mathbf{c} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{a}$

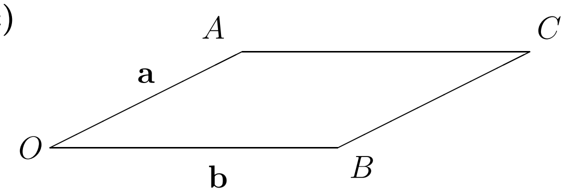
Since HK is perpendicular to BC so $k\mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0$

therefore $\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$

Thus $\mathbf{c} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} = 0$ so $l\mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = 0$

i.e. HL is perpendicular to AC

(c)



$$\begin{aligned} \vec{OC} &= \mathbf{a} + \mathbf{b} \\ \vec{AB} &= \mathbf{b} - \mathbf{a} \\ \vec{OC} \cdot \vec{AB} &= (a + b) \cdot (b - a) \\ &= |b|^2 - |a|^2 = 0 \\ & \quad \text{Since } |OA| = |OB| \end{aligned}$$