Question

The cartesian co-ordinates of the points A, B, C are $(-1, 1, 0)$, $(1, 4, 6)$, $(3, 5, 7)$ respectively. Find

(i) The components of $\vec{AB}$ and $\vec{AC}$.

(ii) The direction cosines of line BC.

(iii) The parametric form of the equation BC and give its cartesian form.

(iv) The parametric form of the equation of the plane π containing A, B, C.

(v) The sines of the angle BAC.

(vi) The components of the unit vector $\hat{\mathbf{n}}$ perpendicular to the plane π such that $\vec{AB}, \vec{AC}, \hat{\mathbf{n}}$ form a right-handed system.

(vii) The 'normal 'form the equation of the plane π and check it agrees with part (iv)

(viii) What is the shortest distance from O to the plane π?

(xi) The shortest distance between the lines BC and OA.

(x) The equation of the line perpendicular to both BC and OA.

Answer

$A = (-1, 1, 0)$, $B = (1, 4, 6)$, $C = (3, 5, 7)$

(i) $\vec{AB} = (2, 3, 6)$

$\vec{AC} = (4, 4, 7)$

(ii) $\vec{BC} = (2, 1, 1)$ So $\hat{BC} = \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

(iii) $\mathbf{r} = (1, 4, 6) + t(2, 1, 1)$

$$\frac{x-1}{2} = \frac{y-4}{1} = \frac{z-6}{1}$$

(iv) $r = \vec{OA} + u\vec{AB} + u\vec{AC} = (-1,1,0) + u(2,3,6) + v(4,4,7)$

$x = -1 + 2 + 4v \quad y = 1 + 3u + 4v \quad z = 6u + 7v$

(v) $\cos BAC = \dfrac{AB \cdot AC}{|AB||AC|} = \dfrac{62}{63}$

(vi) $\vec{AB} \times \vec{AC} = (-3, 10, -4) \quad \hat{n} = \left(-\dfrac{3}{\sqrt{125}}, \dfrac{10}{\sqrt{125}}, -\dfrac{4}{\sqrt{125}}\right)$

(vii) The equation of π is $-3x + 10y - 4z = k$

$(-1,1,0)\epsilon\pi$ So $k = 13$

Check with (iv) $-3(1+2u+4v) + 10(1+3u+4v) - 4(6u+7v) = 13\surd$

(viii) Shortest distance from $\mathbf{p}$ to $\mathbf{r} \cdot \mathbf{a} = k$ is $\left|\dfrac{a \cdot p - k}{|a|}\right|$ so when $p = 0$

$$d = \frac{|k|}{|a|} = \frac{13}{|(-3,10,-4)|} = \frac{13}{\sqrt{125}}$$

(xi) & (x)

The line BC has parametric equation $\mathbf{r} = (1,4,6) + t(2,1,1) \quad - L$

The line OA has parametric equation $\mathbf{r} = (0,0,0) + t(-1,1,0) \quad - M$

Let P, Q be points on L, M

$\vec{QP} = (1 + 2k + l,\ 4 + k - l,\ 6 + k)$

We want $\vec{QP} \cdot (2,1,1) = 0$ and $\vec{QP} \cdot (-1,1,0) = 0$

So $2 + 4k + 2; +4 + k - l + 6 + k = 0 \qquad 6k + l = -12$

$-1 - 2k - l + 4 + k - l = 0 \qquad -k - 2l = -3$

So $4k = -\frac{27}{11} \quad l = \frac{30}{11} \quad P = \left(-\frac{43}{11}, \frac{17}{11}, \frac{39}{11}\right)\ Q = \left(-\frac{30}{11}, \frac{30}{11}, 0\right)$

$\vec{QP} = \frac{13}{11}(-1,-1,3)$ so $|QP| = \frac{13}{11} \cdot \sqrt{11}$

The equation of QP is $\mathbf{r} = \left(-\frac{43}{11}, \frac{17}{11}, \frac{39}{11}\right) + t\left(-\frac{30}{11}, \frac{30}{11}, 0\right)$

$$\frac{x + \frac{30}{11}}{-\frac{13}{11}} = \frac{y - \frac{30}{11}}{-\frac{13}{11}} = \frac{z}{\frac{39}{11}} \quad \text{or} \quad 11x + 30 = 11y - 30 = -\frac{11}{3}z$$