QUESTION

Obtain at least one solution of the form

$$y = x^{\sigma} \sum_{n=0}^{\infty} a_n x^n$$

for each of the following differential equations. Where possible, obtain a second independent solution of the same form, or comment on why it is not possible to do so.

$9x\,(1-x)\,y'' - 12y' + 4y = 0$

ANSWER

$9x\,(1-x)\,y'' - 12y' + 4y = 0$

$y'' - \frac{12}{9x(1-x)} + \frac{4}{9x(1-x)}y = 0.$ 0 is a regular singular point

$-9x^2y'' + 4x + 9xy'' - 12y' = 0$

$\sum_{n=0}^{\infty} a_n\{[-9\,(\sigma+n)\,(\sigma+n-1)+4]\,x^{\sigma+n}$

$+[9\,(\sigma+n)\,(\sigma+n-1) - 12\,(\sigma+n)]\,x^{\sigma+n-1}\} = 0$

Factorizing

$\sum_{n=0}^{\infty} a_n\{-[3\,(\sigma+n)-4]\,[3\,(\sigma+n)+1]\,x^{\sigma+n}$

$+[3\,(\sigma+n)]\,[3\,(\sigma+n)-7]\,x^{\sigma+n-1}\} = 0$

Reordering

$\sum_{n=0}^{\infty} x^{\sigma+n}\{-a_n\,[3\,(\sigma+n)-4]\,[3\,(\sigma+n)+1]$

$+[3\,(\sigma+n)+3]\,[3\,(\sigma+n)-4]\,a_{n+1}\} + a_0x^{\sigma-1}3\sigma\,(3\sigma-7) = 0$

$\sigma = 0$ or $\sigma = \frac{7}{3}$ and $a_{n+1} = \frac{3(\sigma+n)+1}{3(\sigma+n)+3}$

$\sigma = 0:\ \ a_{n+1} = \frac{3n+1}{3(n+1)}a_n$

$a_n = \frac{3n-2}{3n}a_{n-1} = \frac{(3n-2)(3n-5)}{3n\cdot 3(n-1)}a_{n-2} \Rightarrow a_n = \frac{(3n-2)(3n-5)\ldots 1}{3^n n!}a_0$

$\sigma = \frac{7}{3}:\ \ a_{n+1} = \frac{3n+8}{3n+10}a_n$

$a_n = \frac{3n+5}{3n+7}a_{n-1} = \frac{(3n+5)(3n+2)\ldots 8}{(3n+7)(3n+4)\ldots 10}a_0$