QUESTION

Obtain at least one solution of the form

$$y = x^{\sigma}\sum_{n=0}^{\infty} a_n x^n$$

for each of the following differential equations. Where possible, obtain a second independent solution of the same form, or comment on why it is not possible to do so.

$2x(1-x)y'' + (1-x)y' + 3y = 0$

ANSWER

$2x(1-x)y'' + (1-x)y' + 3y = 0$

$y'' + \frac{1}{2x}y' + \frac{3}{2x(1-x)}y = 0$ so 0 is a regular singular point. We group terms together that will give the same powers of x:

$2xy'' + y' - 2x^2y'' - xy' + 3y = 0$

$\sum_{n=0}^{\infty} a_n \Big\{[2(\sigma+n)(\sigma+n-1) + (\sigma+n)]\, x^{\sigma+n-1}$

$+ [-2(\sigma+n)(\sigma+n-1) - (\sigma+n) + 3]\, x^{\sigma+n}\Big\} = 0$

Factorising gives

$\sum_{n=0}^{\infty} a_n\{(\sigma+n)(2\sigma+2n-1)\, x^{\sigma+n-1}$

$-[(2\sigma+2n-3)(\sigma+n+1)\, x^{\sigma+n}] = 0$

Reordering by powers of x gives

$a_0\sigma(2\sigma-1)\, x^{\sigma-1}$

$+\sum_{n=0}^{\infty} x^{\sigma+n}\{a_{n+1}(\sigma+n+1)(2\sigma+2n+1)$

$-(2\sigma+2n-3)(\sigma+n+1)\, a_n\} = 0$

$\sigma(2\sigma-1) = 0 \Rightarrow \sigma = 0$ or $\sigma = \frac{1}{2}$

$a_{n+1} = \frac{2\sigma+2n-3}{2\sigma+2n+1}$

$\sigma = 0:\ \ a_{n+1} = \frac{2n-3}{2n+1}a_n$

$a_n = \frac{2n-5}{2n-1}a_{n-1} = \frac{2n-5}{2n-1}\frac{2n-7}{2n-3}\frac{2n-9}{2n-5}\ldots\frac{-1}{3}\frac{-3}{1}a_0$

$\Rightarrow a_n = \frac{3}{(2n-1)(2n-3)}a_0$ after all cancellations.

$\sigma = \frac{1}{2}:\ \ a_{n+1} = \frac{2n-2}{2n+2}a_n = \frac{n-1}{n+1}a_n$

$a_1 = -a_0,\ a_2 = 0,\ a_3 = 0, \ldots$

General solution is

$$y = A\sum_{n=0}^{\infty}\frac{1}{(2n-1)(2n-3)}x^n + B\left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)$$