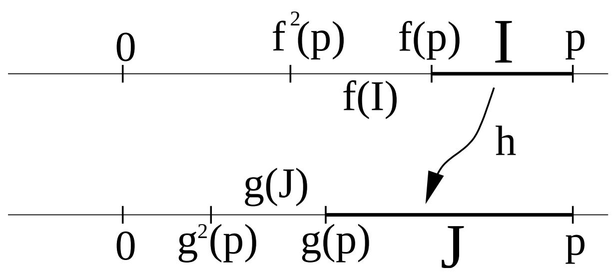
Question

Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ and $g: \mathbf{R} \longrightarrow \mathbf{R}$ be two diffeomorphisms, each having the origin as an attracting fixed point (no flips) with basin of attraction the whole of $\mathbf{R}$. Choose some $p > 0$ and let $I = [f(p), p]$, $J = [g(p), p]$. Construct $h: I \longrightarrow J$ of the form $h(x) = ax + b$ so that $h(p) = p$ and $h(f(p)) = g(p)$. Use this to construct a conjugacy between f, g on $\mathbf{R}$.

Answer



Note: This requires f, g to be invertible.

Consider the intervals $f(I) = [f^2(p), f(p)]$ and $g(J) = [g^2(p), g(p)]$.
Define $h_1: f(I) \longrightarrow g(J)$ by $h_1(x) = g \cdot h \cdot f^{-1}(x)$ (if a conjugacy exists then on $f(I)$ it has to be this.) . Then if $x = f(u)$ (say) where $u \in I$ then $h_1 f(u) = gh(u)$, i.e. $h_1 \circ f = g \circ h : I \longrightarrow g(J)$.
Next define $h_2: f^2(I) \longrightarrow g^2(J)$ by $h_2(x) = g \cdot h_1 \cdot f^{-1}(x)$: this gives $h_2 \circ f = h \circ h_1 : f(I) \longrightarrow g^2(J)$. Continue indefinitely, defining $h_n : f^n(I) \longrightarrow g^n(J)$ by $h_n(x) = g \cdot h_{n-1} f^{-1}(x)$. Likewise define $h_{-1} : f^{-1}(I) \longrightarrow g^n(J)$ by $h_{-1}(x) = g^{-1} h f(x)$, so $g h_{-1} = hf : f^{-1}(I) \longrightarrow J_1$ and inductively define $h_{-n} : f^{-n}(I) \longrightarrow g^{-n}(J)$ by $h_{-n}(x) : g^{-1} \cdot h_{-n+1} f^{-1}(x)$ $(n = 1, 2, 3, \cdots)$. Then (writing $h = h_0$) the family of maps $\{h_m\}_{m \in \mathbf{Z}}$ defines a continuous (both ways) bijection $\mathbf{R}^+ \longrightarrow \mathbf{R}^+$ conjugating f, g. Do likewise for $\mathbf{R}^-$. Finally, map 0 to 0.