**Question**

Find all the fixed points, 2-cycles and find two 3-cycles for the hyperbolic toral automorphism defined by the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

**Answer**

$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$: $(A - I)^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 2 \\ 2 & 0 \end{pmatrix}$;

fixed points $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -2 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix}$ mod 1; $k, l \in \mathbf{Z}$.

Distinct options $k = 0, 1$ and $l = 0, 1$ give

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$.

$A^2 = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$ : $(A^2 - I)^{-1} = \frac{1}{16}\begin{pmatrix} -12 & 8 \\ 8 & -4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$.

The number of fixed points of $f_A^2$ is $|det(A^2 - I)| = 16$; of these, 4 are fixed points of $f_A$ which leaves <u>six</u> 2-cycles.

Taking $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix}$ mod 1 with $k, l \in \mathbf{Z}$ gives factor $\frac{1}{4}$ times:

$$\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}\right\}.$$

$A^3 = \begin{pmatrix} 21 & 34 \\ 34 & 55 \end{pmatrix}$

$$(A^3 - I)^{-1} = \frac{1}{76}\begin{pmatrix} -54 & 34 \\ 34 & -20 \end{pmatrix} = \frac{1}{38}\begin{pmatrix} -27 & 17 \\ 17 & -10 \end{pmatrix}.$$

3-cycles($\times\frac{1}{38}$) $\left\{\begin{pmatrix} -27 \\ 17 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}\right\}, \left\{\begin{pmatrix} -17 \\ 10 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \end{pmatrix}\right\}, \cdots$