Question

Find (a) $\lim_{n\to\infty} \begin{pmatrix} 4.5 & 8 \\ -2 & -3.5 \end{pmatrix}^n \begin{pmatrix} 6 \\ 9 \end{pmatrix}$, (b) $\lim_{n\to\infty} \begin{pmatrix} \frac{3}{2} & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}^n \begin{pmatrix} 6 \\ 9 \end{pmatrix}$

Answer

(a) $A = \begin{pmatrix} 4.5 & 8 \\ -2 & -3.5 \end{pmatrix}$ has eigenvalues: $(\lambda - 4.5)(\lambda + 3.5) + 16 = 0$

i.e. $\lambda^2 - \lambda = 0.25 = 0$ i.e. $\lambda = \frac{1}{2}, \frac{1}{2}$.

These are inside the unit circle, so the origin is a <u>sink</u>. This means <u>$A^n v \to (0,0)$</u> for <u>every</u> vector $v \in \mathbf{R}^2$ (not just (6,9)).

(b) $A = \begin{pmatrix} \frac{3}{2} & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$ has eigenvalues: $(\lambda - \frac{3}{2})\lambda + \frac{1}{2} = 0$

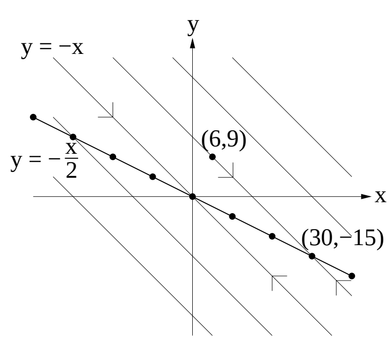
i.e. $\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0 :\ (\lambda - 1)(\lambda - \frac{1}{2}) = 0$. Hence $\lambda = 1, \dfrac{1}{2}$.

This means (0,0) is a non-hyperbolic fixed point.

Eigenvectors:

$\lambda = 1 \quad \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} : \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, say.

$\lambda = \frac{1}{2} \quad \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} : \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, say.



Thus under the action of A, every point on the line $y = -\frac{1}{2}x$ remains fixed (there is a whole line of fixed points), while every vector in the direction $y = -x$ is shrunk by a factor $\dfrac{1}{2}$. Hence the point (6,9) will be attracted to the point where the line through (6,9) with slope -1 meets the line $y = \frac{-x}{2}$. Easy to check this point is (30,-15).