Question

Verify thet, if U is a constant, then the stream function $\phi(x, y) = Uxy$ represents two-dimensional inviscid flwo with no bsy forces in the region $\{(x, y) : (x \geq 0), (y \geq 0)\}$ bounded by solid walls at $x = 0$ and $y = 0$. Derive the (dimensional) boundary layer equations for flow near to the solid wall $y = 0$ in the form

$$\begin{aligned} uu_x + vu_y &= xU^2 + \nu u_{yy} \\ u_x + v_y &= 0 \end{aligned}$$

where the fluid velocity $\underline{q}$ is given by (u, v) and ν denotes the constant kinematic viscosity of the fluid, giving suitable boundary conditions for these equations.

Show that these equations possess a similarity solution of the form

$$\begin{aligned} \phi &= \sqrt{U\nu} x^k f(\mu) \\ \mu &= \sqrt{\frac{Uy^2}{\nu}} \end{aligned}$$

provided k is suitably chosen. Obtain the ordinary differential equation satisfied by f and state boundary conditions that f must satisfy. Show further that, when $\mu \to 0$, $f \sim -\mu^3/6$.

Answer

If $\psi = Uxy$ then $\nabla^2\psi = 0 + 0 = 0$

Also

$$\begin{aligned} u = \psi_y &= Ux \\ v = -\psi_x &= -Uy \end{aligned}$$

so $u = 0$ on $x = 0$ and $v = 0$ on $y = 0$ (nv if they wanted to they could also show that this satisfies the Euler equations with $p = -\frac{\rho U^2}{2}(x^2+y^2)+constant$)

Now use the N/S equations in component form:-

Scale and non-dimensionalise using

$$\begin{aligned} u &= Uu' \\ v &= \delta Uv' \\ x &= Lx' \\ y &= \delta Ly' \\ p &= \rho U_\infty^2 p' \end{aligned}$$

to look in the boundary layer near $y = 0$ where $\delta \ll 1$ is to be found.

$$\Rightarrow \left.\begin{aligned} \frac{U^2}{L}(uu_x + vu_y) &= -\frac{U^2}{L}p_x + \nu\left(\frac{U}{L^2}u_{xx} + \frac{U}{L^2\delta^2}u_{yy}\right) \\ \frac{\delta U^2}{L}(uv_x + vv_y) &= -\frac{U^2}{L\delta}p_y + \nu\left(\frac{\delta U}{L^2}v_x x + \frac{U}{L^2\delta}\right) \\ u_x + v_y &= 0 \end{aligned}\right\}$$

(all bars dropped)

So re-arranging

$$\begin{aligned} uu_x + vu_y &= -p_x + \frac{1}{Re}\left(u_{xx} + \frac{1}{\delta^2}u_{yy}\right) \\ \delta(uv_x + vv_y) &= -\frac{1}{\delta}p_y + \frac{1}{Re}\left(\delta v_{xx} + \frac{1}{\delta}v_{yy}\right) \\ u_x + v_y &= 0 \end{aligned}$$

The only chance for a non-trivial balance that retains a 2nd-order system is to choose $\delta^2 re = O(1)$.

So take $\delta = \frac{1}{\sqrt{Re}}$ $\Rightarrow$ to lowest order

$$\left.\begin{aligned} uu_x + vu_y &= -p_x + u_{yy} \\ 0 &= -p_y \\ u_x + v_y &= 0 \end{aligned}\right) \longrightarrow (1)$$

In the outer flow (dimensionally) $p + \frac{1}{2}\rho \underline{q}^2 = constant$

Now $\underline{q} = (U_x, -U_y)$ so outside the boundary layer (near $y = 0$)

$$\begin{aligned}
\underline{q} &\sim U_x\hat{\underline{e}}_x \\
\Rightarrow \quad p_x + \rho(U_x)U &= 0 \\
\Rightarrow \quad -p_x/\rho &= U^2 x
\end{aligned}$$

So redimensionalising (1) gives

$$\left.\begin{aligned}
uu_x + vu_y &= U^2 x + u_{yy}\nu \\
u_x + v_y &= 0
\end{aligned}\right\}$$

B/C's (No slip) $u = v = 0$ at $y = 0$

(Matching) $u \to U_x$ as $y \to \infty$

Similarity solutions:- use

$$\begin{aligned}
\psi &= \sqrt{U\nu}x^k f\eta \\
\eta &= (Uy^2/\nu)^{\frac{1}{2}} = \sqrt{\frac{U}{\nu}}y
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad u &= Ux^k f' \\
u_y &= U\sqrt{\tfrac{U}{\nu}}x^k f'' \\
u_{yy} &= \frac{U^2}{\nu}f'''x^k \\
v &= -\sqrt{U\nu}kx^{k-1}f \\
u_x &= kUx^{k-1}f'
\end{aligned}$$

$$\begin{aligned}
\Rightarrow Ux^k f'(kUx^{k-1}f') - \sqrt{U\nu}kx^{k-1}f\frac{U^{\frac{3}{2}}}{\sqrt{\nu}}x^k f'' &= U^2 x + \frac{U^2}{\nu}x^k f'''\nu \\
kU^2x^{2k-1}f'^2 - kU^2x^{2k-1}ff'' &= U^2 x + \frac{U^2}{\nu}\nu x^k f'''
\end{aligned}$$

Obviously the similarity solution can only work if $k = 1$, in which case the ODE is

$$U^2 f'^2 - U^2 ff'' = U^2 + U^2 f'''$$

i.e.

$$f''' + ff'' + 1 - f'^2 = 0$$

Suitable B/C's (No slip) $f'(0) = f(0) = 0$

(Matching) $f'(\infty) = 1$

Now consider the ODE for small η. Since for small values of η, f and f' are small, the leading order balance in the equation will be

$$f''' + 1 = 0$$

$$\begin{aligned}
\Rightarrow f'' &\sim -\eta + A \\
f' &\sim -\eta^2/2 + A\eta + B \\
f &\sim -\eta^3/6 + A\eta^2/2 + B\eta + C
\end{aligned}$$

But since $f(0) = f'(0) =)$, $A = B = 0$. Also f is a stream function so we can take $C = 0$

$$\Rightarrow f \sim -\frac{\eta^3}{6} \quad (\eta \sim 0)$$