Question

(i) Using tensorial notation, or otherwise, prove the vector identities

$$\nabla.(\phi\underline{a}) = \phi\nabla.\underline{a} + (\underline{a}.\nabla)\phi$$

$$\nabla \times (\nabla \times \underline{a}) = -\nabla^2\underline{a} + \nabla(\nabla.\underline{a})$$

$$\underline{a} \times (\nabla \times \underline{a}) = \nabla(\underline{a}^2/2) - (\underline{a}.\nabla)\underline{a}$$

for a general suitably differentiable scalar function ϕ and vector $\underline{a}$.

(ii) Explain briefly the major differences between the 'Eulerian' and 'Lagrangian' descriptions of flow. What does it mean to say that a fluid flow is 'steady'? For a fluid flow with velocity $\underline{q}$ define $\underline{\omega}$, the *vorticity*, and give the condition that the flow is *irrotational*.

(iii) YOU MAY ASSUME that the Navier-Stokes equations for an unsteady viscous flow are given by

$$\begin{aligned} \underline{q}_t + (\underline{q}.\nabla)\underline{q} &= -\frac{1}{\rho}\nabla\underline{p} + \nu\nabla^2\underline{q} + \underline{F} \\ \nabla.\underline{q} &= 0 \end{aligned}$$

where ρ denotes the (constant) density, ν the (constant) kinematic viscosity, $\underline{p}$ the pressure and $\underline{F}$ the body force. Show that for an inviscid irrotational flow with a conservative body force the quantity

$$\phi_t + \frac{1}{2}|\underline{q}|^2 + \chi$$

is a function of time alone, identifying the quantities ϕ and χ. Show also that for a steady (but not necessarily irrotational) inviscid flow the quantity

$$\frac{\underline{p}}{\rho} + \frac{1}{2}|\underline{q}|^2 + \chi$$

is constant along both streamlines and vortex lines in the flow.

Answer

(i)

$$\begin{aligned} div(\phi \underline{a}) = \frac{\partial}{\partial a_j}(\phi a_j) &= \frac{\partial \phi}{\partial x_j} a_j + \phi \frac{\partial a_j}{\partial x_j} \\ &= (\underline{a}.\nabla)\phi + \phi div(\underline{a}) \end{aligned}$$

$$\nabla \times (\nabla \times \underline{a}) = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \underline{a})_k = \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{kpq} \frac{\partial}{\partial x_p} a_q$$

$$= \epsilon_{ijk} \epsilon_{kpq} \frac{\partial^2 a_q}{\partial x_j \partial x_p} = \epsilon_{kij} \epsilon_{kpq} \frac{\partial^2 a_q}{\partial x_j \partial x_p} = (\delta_{ip}\delta_{jq} - \delta_{jp}\delta_{iq}) \frac{\partial^2 a_q}{\partial x_j \partial x_p}$$

$$= \frac{\partial^2}{\partial x_j \partial x_i} - \frac{\partial^2 a_i}{\partial x_j \partial x_j} = \nabla(div(\underline{a})) - \nabla^2 \underline{a}$$

$$\underline{a} \times (\nabla \times \underline{a}) = \epsilon ijk a_j (\nabla \times \underline{a})_k = \epsilon_{ijk} a_j \epsilon_{kpq} \frac{\partial a_q}{\partial x_p}$$

$$= \epsilon_{kij} \epsilon_{kpq} a_j \frac{\partial a_q}{\partial x_p} = (\delta_{ip}\delta_{jq} - \delta_{jp}\delta_{iq}) a_j \frac{\partial a_q}{\partial x_p}$$

$$= a_j \frac{\partial a_j}{\partial x_i} - a_j \frac{\partial a_i}{\partial x_j} = \nabla(\frac{1}{2}\underline{a}^2) - (\underline{a}\nabla)\underline{a}$$

(ii) EULERIAN: We try to find the fluid velocity $\underline{q}$ as a function of $\underline{x}$ and t. Attention is focussed on a particular position in the flow rather than on a particular fluid particle.

LAGRANGIAN: We try to find the fluid motion $\underline{x}$ as a function of $\underline{X}$ and t, where $\underline{X}$ denotes the positions of fluid particles at $t = 0$. Attention is focussed on a given fluid particle following the flow.

A flow is STEADY if $\underline{q}$ does not depend on t.

If a flow has velocity $\underline{q}$ then $\underline{\omega} = curl(\underline{q})$

For IRROTATIONAL FLOW $\underline{\omega} = 0$

(iii) We have

$$\underline{q}_t + (\underline{q}.\nabla)\underline{q} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{q} + \underline{F}, \quad \nabla.\underline{q} = 0$$

For irrotational flow, define $\underline{q} = \nabla\phi$ and for a conservative body force define $\underline{F} = -\nabla\psi$. Then

$(\nabla\phi)_t + (\underline{q}.\nabla)\underline{q} = -\nabla\left(\frac{p}{\rho}\right) - \nabla\psi$ since the fluid is inviscid and ρ is constant.

$\Rightarrow (\nabla\phi)_t + \nabla\left(\frac{\underline{q}^2}{2}\right) = \underline{q} \times (\nabla \times \underline{q}) + \nabla\left(\frac{p}{\rho}\right) + \nabla(\psi) = 0$

But $\nabla \times \underline{q} = 0$ so

$\nabla(\phi_t + \frac{1}{2}\underline{q}^2 + \frac{p}{\rho} + \psi) = 0$. The quantity in brackets is independent of x, y and z and thus

$$\phi_t + \frac{1}{2}|\underline{q}|^2 + \frac{p}{\rho} + \psi = f(t) \text{ only}$$

(ϕ =velocity potential, ψ = force potential)

For a steady inviscid flow $(\underline{q}.\nabla)\underline{q} = -\nabla(p/\rho) - \nabla\psi$

$\Rightarrow \nabla(\underline{q}^2/2) - \underline{q} \times (\nabla \times \underline{q}) = -\nabla(p/\rho) - \nabla(\psi)$

$\Rightarrow \underline{q} \times \underline{\omega} = \nabla(\underline{q}^2/2 + p/\rho + \psi)$

Thence $\underline{q}.(\underline{q} \times \underline{\omega}) = 0 = (\underline{q}.\nabla)(\underline{q}^2/2 + p/\rho + \psi)$ and so $p/\rho + \frac{1}{2}|\underline{q}|^2 + \psi$ is constant along streamlines in the flow.

Also $\underline{\omega}.(q \times \underline{\omega}) = 0$ and so

$\underline{q}^2/2 + p/\rho + \psi$ is constant along vortex lines in the flow.