QUESTION

2	7	6
9	5	1
4	3	8

Figure 1
($n = 3$)

0	0	c	d
0	0	g	
i			

figure 2
($n = 4$)

A magic square is an $n \times n$ matrix containing the numbers $1, 2, \dots n^2$ with the property that the sum of the entries along each row, each column and each of the two main diagonals is the same. (See figure 1 for an example with $n = 3$.) The idea can be generalised to squares containing any n^2 numbers (not necessarily distinct). Show that there is a unique way to fill in the remaining squares in figure 2 to make it into a magic square.

ANSWER

From the first row the magic constant is $c + d$. This can be used to find the following entries: $(M)_{24}, (M)_{41}, (M)_{32}, (M)_{42}$ to give a square of the form

$$\begin{bmatrix} 0 & 0 & c & d \\ 0 & 0 & g & c+d-g \\ i & i-g-d & w & x \\ c+d-i & c+2d+g-i & y & z \end{bmatrix}$$

The magic condition on rows three and four, columns three and four and the main diagonal lead to five equations (which prove to be consistent) in the four unknowns w, x, y and z and a sixth equation can also be obtained from the fact that the sum of the entries in the last two columns (or rows) is twice the magic constant. Hence one has a choice of equations to solve for w, x, y and z:

$$\begin{aligned} w + x &= c + 2d + g - 2i && (1) \\ w + y &= d - g && (2) \\ w + z &= c + d && (3) \\ x + z &= -d + g && (4) \\ y + z &= -c - 2d - g + 2i && (5) \\ w + x + y + z &= 0 && (6) \end{aligned}$$

[This system has a unique solution for w, x, y and z if and only if precisely four of the equations are linearly independent. But

$(5) = -(1) + (2) + (4)$ and $(6) = (2) + (4)$.
Thus (5) and (6) are not needed to solve the equations, but the four equations $(1), (2), (3), (4)$ are independent, so the system does have a unique solution.]
The solution is

$$\begin{bmatrix} 0 & 0 & c & d \\ 0 & 0 & g & c+d-g \\ i & i-g-d & c+2d-i & g-i \\ c+d-i & c+2d+g-i & -c-d-g+i & -d+i \end{bmatrix}$$