Question

What sort of geometric shapes are the objects $|z-9i| \leq 15$, $|z+16i| \leq 20$? Where do the boundaries of these regions meet the real axis? Sketch the regions. Show that the Möbius map $w = \left(\dfrac{z-12}{z+12}\right)$ transforms the boundaries into two straight lines, and find the equations of these lines.

Answer

$$\left.\begin{array}{rcl} |z-9i| & \leq & 15 \\ |z+16i| & \leq & 20 \end{array}\right\} \text{are filled circles}$$

(1) $|z-9i| < 15 \Rightarrow$ circle centre $9i$, radius ≤ 15

(2) $|z+16i| \leq 20 \Rightarrow$ circle centre $-16i$, radius ≤ 20

(z)

PICTURE

(1) intersects x-axis where

$$\begin{aligned} |x-9i| &= 15 \\ \Rightarrow x^2+81 &= 225 \\ \Rightarrow x^2 &= 144 \\ x &= \pm 12! \end{aligned}$$

(2) intersects x-axis where

$$\begin{aligned} |x+16i| &= 20 \\ \Rightarrow x^2+256 &= 400 \\ \Rightarrow x^2 &= 144 \\ x &= \pm 12! \end{aligned}$$

If we want to map circles $\rightarrow$ lines we need something to go to ∞ in the Möbius mapping.
$w = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, e.g., we could map $z = -12$ to ∞. Then any circle through $z = -12$ maps to a line, so take $w = \dfrac{z - 12}{z + 12}$ say.
What lines?
$w(z + 12) = (z - 12) \Rightarrow z = -12\dfrac{(w + 1)}{(w - 1)}$
so

$$
\begin{aligned}
|z - 9i| = 15 \quad &\rightarrow \quad \left| -12\frac{(w+1}{(w-1)} - 9i \right| = 15 \\
&\Rightarrow \quad \left| w + \frac{(12 - 9i)}{(12 + 9i)} \right| = |w - 1| \\
|z + 16i| = 20 \quad &\rightarrow \quad \left| -12\frac{(w+1}{(w-1)} + 16i \right| = 20 \\
&\Rightarrow \quad \left| w + \frac{(12 + 16i)}{(12 - 16i)} \right| = |w - 1|
\end{aligned}
$$

UGH!
Note that $\left(\dfrac{12 - 9i}{12 + 9i}\right)$ and $\left(\dfrac{12 + 16i}{12 - 16i}\right)$ lie on the circle $|w| = 1$. Thus the lined bisect the chords joining two points on a circle's circumference. Thus the two lined must pass through the centre of $w = 0$!
PICTURE

A, B are any two points on circle centre 0. The bisector of chord AB is what we want, which by definition (equals θ's) must pass through 0!