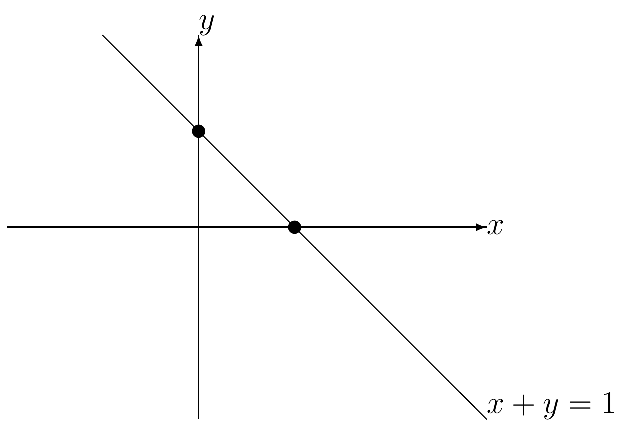
Question

Show that the line $Re(z) + Im(z) = 1$ can be written as $|z| = |z - 1 - i|$. Hence, by building up the mapping in a series of steps, find a transformation which takes this line to the unit circle $|w| = 1$.

Answer

Plot the region boundary:
$z = x + iy$
$Re(z) + (z) < 1 \Rightarrow x + y < 1$



If we can write it as $|z| = |z - (1 + i)|$, then the distance of any z (on the line $x + y = 1$) from $(1 + i)$ and 0 is the same. Thus we have to show that $x + y = 1$ bisects the line from 0 to $(1 + i)$. Simple geometry comes into play:
Let OPQ lie on $y = x$ with $|OP| = |PQ| \longrightarrow P$ is then $(x, y) = (\frac{1}{2}, \frac{1}{2})$
$\overline{OP} = (\frac{1}{2}, \frac{1}{2})$
$\Rightarrow \overline{PQ} = (1, 1)$
Thus $Q = \underline{1 + i}$
PICTURE

Hence
$|z-0| = |z-(1+i)|$
$\Rightarrow |z| = |z-(1+i)|$ as required.
Now to map line $\longrightarrow$ a circle.

(i) Try an inversion $w_1 = \dfrac{1}{z}$. Why?

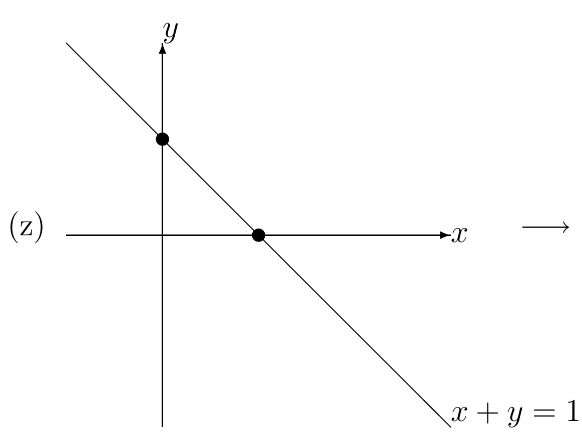
$|z| = \infty \xrightarrow{w_1} w_1 = 0$ finite

all other z on line $x+y=1$ are finite and $\neq 0 \xrightarrow{w_1} w_1 = 0$ finite

We know that inversions $1z$ map lines/circles to lines/circles, but all points in w, which are images of $x+y=1$ points are finite. Hence image of $x+y=1$ <u>must</u> be a circle.

$$
\begin{aligned}
w_1 = \frac{1}{z} \quad \Rightarrow \quad \left|\frac{1}{w_1}\right| &= \left|\frac{1}{w_1} - (1+i)\right| \\
\Rightarrow \quad 1 &= |1-(1+i)w_1| \\
\text{or} \quad 1 &= |w_1(1+i)-1| \\
\text{or} \quad \frac{1}{|1+i|} &= \left|w_1 - \frac{1}{1+i}\right| \\
\Rightarrow \quad \frac{1}{\sqrt{2}} &= \left|w - 1 - \frac{1}{1+i}\right|
\end{aligned}
$$

i.e., distance from $1+i$ in w_1 plane is always $\dfrac{1}{\sqrt{2}} \Rightarrow$ a circle centre $w_1 = \dfrac{1}{1+i}$, radius $\dfrac{1}{\sqrt{2}}$.



$\longrightarrow$ PICTURE

(ii) Shift circle to origin $w_2 = w_1 - \dfrac{1}{1+i}$

PICTURE

(iii) Scale radius by $\sqrt{2}$ $w_3 = \sqrt{2}w_2$.

PICTURE

Set $w_3 = w$ and assemble (i) $\rightarrow$ (iii).

$w = \sqrt{2}w_2 = \sqrt{2}\left(w_1 - \dfrac{1}{1+i}\right) = \sqrt{2}\left(\dfrac{1}{z} - \dfrac{1}{1+i}\right)$

$\Rightarrow w = \sqrt{2}\dfrac{((1+i)-z)}{(1+i)z}$

Note that $z = 0 \longrightarrow w = \infty$.