Question

The boundary of a domain D_1 in the $z = x+iy$ domain is the ellipse $\left(\frac{X}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = \frac{1}{4}$ together with the line $y = 0$, $|x| < 2$. Show that an application of the Joukowski transform $z = w + \frac{1}{w}$, where $w = u + iv$ maps D_1 onto two concentric circles in the w-plane, centred on the origin, of radius 1 and 2. (Hint: Note that the notation of the Joukowski transform is here the reverse of the lectures. Start with the w-circles and work back to the z-plane).

Answer

The Joukowski tranform $z = w + \frac{1}{w}$ maps w-circles to z-ellipses. (NB notation is opposite of lecture.)

So if $z = w + \frac{1}{w}$ then the circle $|w| = 1$ corresponds to the segment $y = 0$, $|x| < 2$ in the z-plane, as shown in the lectures:

$$\left.\begin{array}{rcl} x & = & \left(R + \frac{1}{R}\right)\cos\theta \\ y & = & \left(R - \frac{1}{R}\right)\sin\theta \end{array}\right\} \xrightarrow{R=1} \left\{\begin{array}{rcl} x & = & 2\cos\theta \\ y & = & 0 \end{array}\right.$$

The circle $|w| = R$ maps to the ellipse

$$\left(\frac{x}{R + \frac{1}{R}}\right)^2 + \left(\frac{y}{R - \frac{1}{R}}\right)^2 = 1$$

To get the given ellipse, we need $R + \frac{1}{R} = \frac{5}{2}$, $R - \frac{1}{R} = \frac{3}{2} \Rightarrow R = 2$, as required.