Question

The region R is the rectangular region in the $z = x + iy$ plane which is bounded by $x = 0,\ y = 0,\ x = 2,\ y = 1$. Determine the region R' of the w plane into which R is mapped under the following transformations.

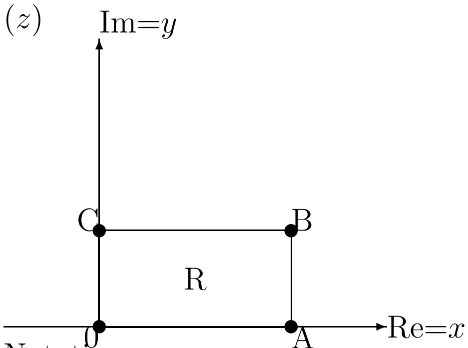
(i) $w = z + (1 - 2i)$

(ii) $w = \sqrt{2} exp\left(\dfrac{i\pi}{4}\right) z$

(iii) $w = \sqrt{2} exp\left(\dfrac{i\pi}{4}\right) z + (1 - 2i)$

Answer

(z)

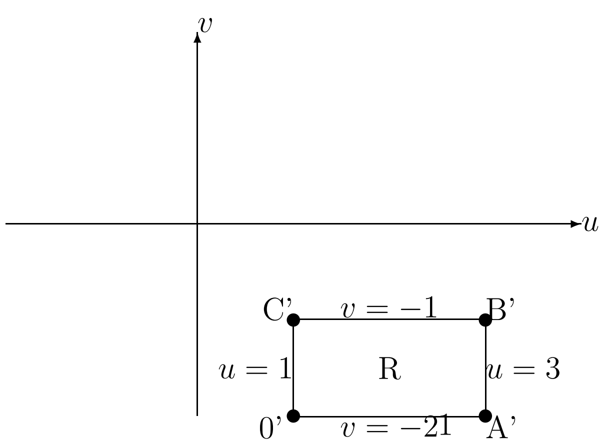


Notation:

Let $0ABC \to 0'A'B'C'$ respectively

(i) $w = z + (1 - 2i)$ is a simple translation by $1 - 2i$.

(w)



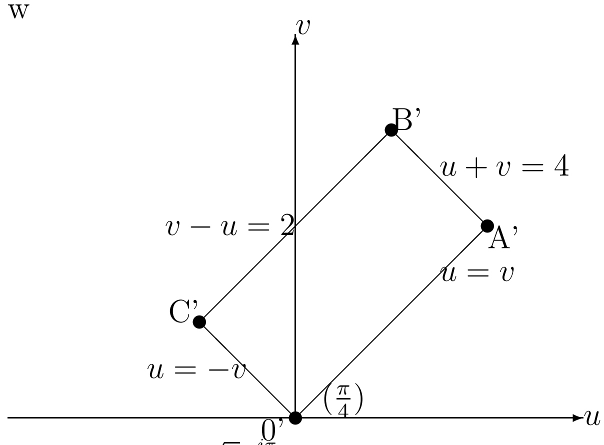
Let $w = u + iv$.

So R' is a region bounded by

$$\left\{ \begin{array}{rcl} u & = & 1 \\ v & = & -1 \\ u & = & 3 \\ v & = & -2 \end{array} \right\}$$

(ii) $w = \sqrt{''} e^{\frac{i\pi}{4}} z$ is a rotation of R by $+\frac{\pi}{4}$, followed by a stretching of $\sqrt{2}$:

w



$w = u + iv = \sqrt{2} e^{\frac{i\pi}{4}} (x + iy) = (1 + i)(x + iy)$

$$\Rightarrow \left\{ \begin{array}{rcl} u & = & x - y \\ v & = & x + y \end{array} \right.$$

Hence

$$x = 0 \rightarrow \left\{ \begin{array}{rcl} u & = & -y \\ v & = & y \end{array} \right\} \Rightarrow u = -v$$

$$y = 0 \rightarrow \left\{ \begin{array}{rcl} u & = & x \\ v & = & x \end{array} \right\} \Rightarrow u = v$$

$$x = 2 \rightarrow \left\{ \begin{array}{rcl} u & = & 2 - y \\ v & = & 2 + y \end{array} \right\} \Rightarrow u + v = 2$$

$$y = 1 \rightarrow \underbrace{\left\{ \begin{array}{rcl} u & = & x - 1 \\ v & = & x + 1 \end{array} \right\}}_{\text{These parametrise curves in the } w \text{ plane}} \Rightarrow \underbrace{v - u = 2}_{\text{eliminate parameters}}$$

In general $w = \alpha z$ accomplishes a rotation and stretching of a region.

(iii) $w = \sqrt{2}e^{\frac{i\pi}{4}}z + (1-2i)$ is a rotation of R by $+\dfrac{\pi}{4}$, followed by a stretching of $\sqrt{2}$, followed by a translation of $1-2i$.

Thus

$u + iv = (1+i)(x+iy) + 1 - 2i$

$$\Rightarrow \begin{cases} u &= x - y + 1 \\ v &= x + y - 2 \end{cases}$$

Therefore

$$x = 0 \rightarrow \left\{ \begin{array}{lcl} u &=& -y+1 \\ v &=& y-2 \end{array} \right\} \Rightarrow u + v = -1$$

$$y = 0 \rightarrow \left\{ \begin{array}{lcl} u &=& x+1 \\ v &=& x-2 \end{array} \right\} \Rightarrow u - v = 3$$

$$x = 2 \rightarrow \left\{ \begin{array}{lcl} u &=& 3-y \\ v &=& y \end{array} \right\} \Rightarrow u + v = 3$$

$$y = 1 \rightarrow \left\{ \begin{array}{lcl} u &=& x \\ v &=& x-1 \end{array} \right\} \Rightarrow u - v = 1$$

(w)