Question Find a reduction formula for $I_n = \int x^n \exp(-x)\, dx$ and evaluate I_3.

Answer $I_n = \int x^n e^{-x}\, dx \quad (\star\star)$

Integrate by parts to reduce the power of n.

$$\begin{array}{rl} u = x^n & \dfrac{dv}{dx} = e^{-x} \\ \dfrac{du}{dx} = nx^{n-1} & v = -e^{-x} \end{array}$$

Therefore

$I_n = -x^n e^{-x} - \int nx^{n-1} x(-e^{-x})\, dx$

$I_n = -x^n e^{-x} + n \underbrace{\int x^{n-1} e^{-x}\, dx}$

but this is I_{n-1} by comparison with $(\star\star)$.

Therefore $\underline{I_n = -x^n e^{-x} + nI_{n-1}}$ is the reduction formula.

Use this reduction formula with $n = 3$:

$$\left.\begin{array}{rcl} I_3 & = & -x^3 e^{-x} + 3I_2 \\ I_2 & = & -x^2 e^{-x} + 2I_1 \\ I_1 & = & -xe^{-x} + I_0 \end{array}\right\} \text{What's } I_0?$$

$I_0 = \int e^{-x}\, dx = -e^{-x} + c$ where c is an arbitrary constant.

Thus recursively substituting $I_1 \to I_2 \to I_3$

$I_3 = -x^3 e^{-x} + 3(-x^2 e^{-x} + 2[-xe^{-x} + \{-e^{-x} + c\}])$

PICTURE

$$\begin{array}{rcl} I_3 & = & -e^{-x}x^3 - 3x^2 e^{-x} + 6[-xe^{-x} - e^{-x} + c] \\ & = & -e^{-x}x^3 - 3x^2 e^{-x} - 6e^{-x} + 6c \\ & = & -e^{-x}[x^3 + 3x^2 + 6x + 6] + 6c \\ & = & \underline{-e^{-x}[x^3 + 3x^2 + 6x + 6] + c'} \end{array}$$

Where c' is a new arbitrary constant.