Question

Decide if these subsets of $\mathbf{R}^2$ are open, closed or neither:

(i) $\{(x_1, x_2) : |x_1| \leq 3\}$

(ii) $\{(x_1, x_2) : |x_1| < 3, x_2 > 5\}$

(iii) $\{(x_1, x_2) : x_1 = 0, x_2 < 0\}$

(iv) $\{(x_1, x_2) : x_1^2 + x_1 x_2^2 > 1\}$.

Answer

(i) Closed: any point x in the complement has $|x_1| > 3$, and then any x' sufficiently close to x (e.g. within δ of x, where $\delta = |x_1| - 3$) has $|x_1| > 3$, so the complement is an open set.

(ii) Open: any x' with $|x - x'| < \delta$ will lie in this set if x does and $\delta > 0$ is chosen sufficiently small (depending on x). [How small?]

(iii) Neither: no point in this set has a δ-nhd contained in the set. Moreover, $(0, 0)$ is the complement but no nhd of $(0, 0)$ is.

(iv) Open: no need to plot contours etc - just observe that (roughly) if x doesn't change much then nor does $x_1^2 + x_1 x_2^2 \cdots$.

More precisely, let $y = x + h$: then

$$(y_1^2 + y_1 y_2^2) - (x_1^2 + x_1 x_2^2) = 2x_1 h_1 + h_1^2 + x_1(2x_2 h_2 + h_2^2) + h_1(x_2^2 + 2x_2 h_2 + h_2^2)$$

and the RHS can be made as small as we please (given x) by taking h small enough. Hence if $x_1^2 + x_1 x_2^2 > 1$ then also $y_1^2 + y_1 y_2^2 > 1$ for all y with $y \in B_\delta(x)$, some $\delta > 0$.

[Remark: This example simply reflects the fact that every polynomial is continuous.]