REAL ANALYSIS
UNIFORM CONTINUITY

Definition $f(x)$ is said to be uniformly continuous in a set $s \Leftrightarrow$, given $\varepsilon > 0 \exists \delta = \delta(\varepsilon) | |f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$ and $x_1, x_2 \in S$.

Theorem Suppose $f(x)$ is continuous in $[a, b]$ relative to $[a, b]$, then $f(x)$ is uniformly continuous in $[a, b]$.

Note: False for open intervals. To prove uniform continuity it suffices to prove the following: Given $\varepsilon > 0 \exists$ a finite subdivision of $[a, b]|$ oscillation $f(x) < \epsilon \ x \in [x_{\nu-1} x_\nu]$

or: if $M_\nu, \ m_\nu$ are the upper and lower bounds of $f(x)$ in $[x_{\nu-1} x_\nu]$, we have

$$M_\nu - m_\nu < \varepsilon \quad \nu = 1, 2, \ldots, n$$

First Proof (using Bisection)

Suppose the result is false. Then $\exists \varepsilon_0$ so that for this ε_0, there is no subdivision of the required type.

We call an interval a good interval if $bd - \underline{bd} < \varepsilon_0$, and bad otherwise.

Subdivide $[a \ b]$ into two equal closed intervals $\left[a, \frac{a+b}{2}\right], \ \left[\frac{a+2}{2}, b\right]$.

At least one of these is bad.

We define J_1 to be the bad interval if there is only one, or the left hand one if there is a choice. Now subdivide J_1 into two equal intervals as before, and again define J_2 to be the bad (or left hand) subinterval of J_1. Continue this process (which cannot terminate). We obtain a sequence of bad intervals $J_1 = [a_1 \ b_1], \ J_2 = [a_2 \ b_2] \ldots$ where $|a_n - b_n| = \frac{b-a}{2^n}$.

Now $a_1 \leq a_2 \leq \ldots \leq b$

Hence $\exists l \in [a, b] | a_n \to l$; and $b_n \to L$ as $n \to \infty$.

But $f(x)$ is continuous at l relative to $[a \ b]$. Hence $\exists \delta > 0$, such that, if we write $I_\delta = [l - \delta, l + \delta] \cap [a \ b]$ and $m_\delta = \underline{bd}_{x \in I_\delta} f \ M_\delta = \overline{bd}_{x \in I_\delta} f(x)$, we have $M_\delta - m_\delta < \varepsilon_0$.

But $\exists N | J_N$ is contained in I_δ. This gives a contradiction since I_δ is good, J_N is bad.

Second Proof Given $\varepsilon > 0$. Consider any pint x in $[a \ b]$.

$\exists \delta | |f(y) - f(x)| < \frac{1}{2}\varepsilon$ whenever $y \in (x - \delta, x + \delta) - 1$.

Let us define $f(y) = f(a)$ for $y < a$ and $f(y) = f(b)$ for $y > b$. Then (1) defines a covering of $[a\ b]$.

By the Heine Borel theorem we con find a finite covering subset S of open intervals.

$$I_\nu = (x_\nu\ y_\nu)\ \ \nu = 1, \ldots, n\ \ x_\nu < y_\nu$$

If we take all the points $x_\nu\ y_\nu$ ($\nu = 1, \ldots, n$ and select those which lie in $[a\ b]$, together with a and b, they define a finite subdivision of $[a\ b],\ a = t_0 < t_1 < \ldots < t_m = b$. Consider any point in one of these intervals of the subdivision. Each interval of the subdivision is covered by one interval of S defined by 1. Therefore in this interval, $[t_{\nu-1}\ t_n u]$ which is covered by $(x_\mu\ y)\mu)\{x_\mu < t_{\nu-1} < t_\nu < v_\mu\}\ |f(x) - f(y)| < \frac{1}{2}\varepsilon$ for all x in the interval therefore $|M_\nu - f(y)| < \frac{1}{2}\varepsilon$ and $|m_\nu - f(y)| < \frac{1}{2}\varepsilon$ therefore

$$M_\nu - m_\nu < \varepsilon\ \ \nu = 1, 2, \ldots, n.$$

Hence the result.

Third Proof Suppose false. Then $\exists \varepsilon_0 > 0|$ there is no δ of the required type.

Choose 2 points $x_1 y_1 ||x_1 - y_1| \leq 1$ and $|f(x_1) - f(y_1)| \geq \varepsilon_0$

Choose 2 points $x_2 y_2 ||x_2 - y_2| \leq \frac{1}{2}$ and $|f(x_2) - f(y_2)| \geq \varepsilon_0$

$\vdots$

Choose 2 points $x_n y_n ||x_n - y_n| \leq \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$

The choice is always possible, for otherwise $\frac{1}{n}$ would be a possible δ.

The sequence $x_1, x_2, x_3 \ldots$ is bounded since each $x_n \in [a\ b]$. Hence it contains a convergent subsequence $x_{r_1}, x_{r_2} \ldots \to l$ as $n \to \infty$ where $l \in [a\ b]\ y_{r_1}, y_{r_2} \ldots \to l$ as $x_{r_1} - y_r \to 0$ as $r \to \infty$.

Since $f(x)$ is continuous $\exists$ an interval I about l—for all x in $I \cap [a\ b] = J,\ |f(x) - f(l)| < \frac{1}{2}\varepsilon_0$.

But $\exists N | x_N$ and $y_N \in J$

$$|f(x_N) - f(l)| < \frac{1}{2}\varepsilon_0\ |f(y_N) - f(l)| < \frac{1}{2}\varepsilon_0 \Rightarrow |f(x_N) - f(Y_N)| < \varepsilon_0$$

Which is a contradiction hence the result.

We can use Uniform Continuity to prove that a continuous function is Riemann integrable. We can find a subdivision $\triangle$ such that each interval $M_\nu - m_\nu < \frac{\varepsilon}{b-a}$, since $f(x)$ is uniformly continuous. Then

$$\sum_{\nu=1}^{n} \delta_\nu (M_\nu - m_\nu) < \sum_{\nu=1}^{n} \delta_\nu \frac{\varepsilon}{b-a} = \varepsilon$$

Therefore

$$S_\triangle - s_\triangle < \varepsilon.$$