ANALYSIS
REAL VARIABLE
SEQUENCES

A monotonic increasing sequence either converges to a finite limit or diverges to $= \infty$.

Proof Let S be the set of numbers in the sequence $\{a_n\}$.

Case (i) S has no upper bound.

Given any $X, \exists n_0 / \ a_{n_0} > X \Rightarrow a_n > X$ for $n \geq n_0$.

This means that the sequence diverges to $+\infty$.

Case (ii) S has an upper bound; $\overline{bd} a_n = a$.

Given $\varepsilon > 0 \exists n_1 / \ a_{n_1} > a - \varepsilon \Rightarrow a - \varepsilon < a_n \leq a$ for $n \geq n_1$.

This means that the sequence converges to a.

Upper and lower limits

Definition

$$\left.\begin{array}{l}\lim_{n\to\infty} sup\ a_n \\ \overline{\lim}_{n\to\infty} a_n\end{array}\right\} = \lim_{n\to\infty} sup_{m\geq n} a_m$$

$$\left.\begin{array}{l}\lim_{n\to\infty} inf\ a_n \\ \underline{\lim}_{n\to\infty} a_n\end{array}\right\} = \lim_{n\to\infty} inf_{m\geq n} a_m$$

Justification (for lim sup) Case (i) Suppose S has no upper bound.

i.e $sup_{m\geq 1} a_m = +\infty$

Then $sup_{m\geq n} a_m = +\infty$ for all n.

In these circumstances we say that $\overline{\lim_{n\to\infty}} a_n = +\infty$

Case (ii) Now suppose S has an upper bound. Write $sup_{m\geq n} a_m = A^{(n)}$

$A^{(1)} A^{(2)} \ldots$ is a monotonic decreasing sequence. If this sequence is bounded below, then it converges to a number Λ, and $\overline{\lim}_{n\to\infty} a_n = \Lambda$, and if not, $\overline{\lim}_{n\to\infty} a_n = -\infty$

Subject to the conventions we have introduced $\overline{\lim} a_n$ always exists.

Suppose $\overline{\lim} a_n = \Lambda$ and suppose Λ finite. Then Λ has the following property (P)

(P) For every $\varepsilon > 0$

(i) $a_n > \Lambda - \varepsilon$ for an infinity of n

(ii) $a_n > \Lambda + \varepsilon$ for at most a finite number of n.

Proof **(i)** If $\exists$ only a finite number of $n/\ a_n > \Lambda - \varepsilon$ then $\Rightarrow \overline{\lim} a_n \leq \Lambda - \varepsilon$.

(ii) If $\exists$ an infinity of $n/\ a_n > \Lambda + \varepsilon \Rightarrow \overline{\lim} a_n \geq \Lambda + \varepsilon$

We may define Λ as the upper bound of the set of limit points of $\{a_n\}$.

Theorem 1

$$\underline{\lim} a_n \leq \overline{\lim} a_n$$

Proof

$$inf_{m \geq n} a_m \leq sup_{m \geq n} a_m$$

Let $n \to \infty$ and the result follows.

Theorem 2

$$\lim_{n\to\infty} a_n = a \Leftrightarrow \underline{\lim}_{n\to\infty} a_n = \overline{\lim_{n\to\infty}} a_n = a$$

Proof **(1)** Suppose that $\lim_{n\to\infty} a_n = a$

Then given $\varepsilon > 0, \exists N = N(\varepsilon)/\ |a_m - a| < \varepsilon$ for $m \geq n$. Hence foe $n \geq N$

$$a - \varepsilon < inf_{m \geq n} a_m \leq sup_{m \geq n} a_m < a + \varepsilon$$

Hence $a - \varepsilon < \lambda \leq \Lambda < a + \varepsilon < a + \varepsilon \Rightarrow \lambda = \Lambda$

(2) Suppose $\underline{\lim}_{n\to\infty} a_n = \overline{\lim_{n\to\infty}} a_n = a$

Then, given $\varepsilon > 0\ \exists N = N(\varepsilon)/$ for $n \geq N$,

$$a - \varepsilon \leq inf_{m \geq n} a_m \leq sup_{m \geq n} a_m \leq a + \varepsilon$$

Hence for $m \geq N$ we have

$$|a_m - a| \leq \varepsilon$$

Theorem 3 It is possible to choose a subsequence $b_1, b_1 \ldots$ from $(a_1, a_2 \ldots, / \lim_{n\to\infty} b_n$ exists $= \overline{\lim_{n\to\infty}} a_n$

Proof **(i)** $\Lambda = +\infty$

(ii) $\Lambda = -\infty$

(iii) Λ finite

Let I_n be the interval $[\Lambda - \frac{1}{n}, \Lambda + \frac{1}{n}]$

¿From property (P);

$\exists r_1 / \ a_{r_1} \in I_1$ Write $b_1 = a_{r_1}$

$\exists r_2 > r_1 / \ a_{r_2} \in I_2$ Write $b_2 = a_{r_2}$

Continue the process, and we obtain a subsequence of the required type.

Corollary A bounded sequence of points in n-dimensional Euclidean space contains a convergent subsequence (In particular the result is true for sequences of real and complex numbers $[R_1 \& R_2]$

Example Every real sequence contains either a strictly monotonic increasing subsequence, or a weakly monotonic decreasing subsequence.

Theorem 4 The General (Cauchy) Principle of Convergence The following condition (c) is a necessary and sufficient condition for $\lim_{n\to\infty} a_n$ to exist.

(c) Given $\varepsilon > 0 \exists N = N(\varepsilon) / \ |a_m - a_n| < \varepsilon$ whenever $m > n \geq N$.

Proof Necessity Suppose $\lim_{n\to\infty} a_n = a$

$\exists N / \ |a_s - a| < \frac{1}{2}\varepsilon$ for $s \geq N$ therefore for $m > n > N$

$$|a_m - a_n| \leq |a_m - a| + |a - a_n| < \varepsilon$$

Sufficiency First Proof.

(i) We prove the sequence bounded.
For $m > n \geq N(1)$ we have

$$|a_m - a_n| < 1$$

Choose a fixed $n_1 \geq N(1)$. Then $|a_m| < |a_{n_1}| + 1$ for $m \geq n_1$.
Therefore $\{a_m\}$ is bounded.

(ii) We show that, given $\varepsilon > 0$,

$$\overline{\lim} a_n - \underline{\lim} a_n < \varepsilon.$$

Choose $N = N(\varepsilon) / \ |a_m - a_n| < \varepsilon$ for $m > n \geq N$.
Then for any $n \geq N$

$$sup_{u\geq n} a_u - inf_{v \geq n} a_v < \varepsilon$$

Letting $n \to \infty$ we have

$$\Lambda - \lambda \leq \varepsilon$$

therefore

$$\Lambda = \lambda$$

Second Proof

(i) The sequence is bounded as before.

(ii) $\exists$ a convergent subsequence $\{a_{n_r}\}$ converging to a, say.

(iii) We now show that $\{a_n\}$ converges to a.
Given $\varepsilon > 0 \exists N = M(\varepsilon)$ such that
(I) $|a - a_{n_r}| < \frac{1}{2}\varepsilon$ for $n_r \geq M$
(II) $|a_m - a_n| < \frac{1}{2}\varepsilon$ for $m > n \geq M$
Let r^* be the least r for which $n_{r^*} \geq M$.
Then for $m > n_{r^*}$ we have

$$|a_m - a| \leq |a_m - a_{n_{r^*}}| + |a_{n_{r^*}} - a| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$$

Therefore the sequence is convergent.
The above theorem applies to complex numbers and to n-dimensional Euclidean space.

Theorem 5 $\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow$ (c) given $\varepsilon > 0 \exists N = N(\varepsilon) / \; |\sum_{r=n}^{m} a_r| < \varepsilon$ whenever $m > n \geq N$

Proof Apply theorem 4 to the sequence of partial sums and the result follows.