Question

Determine whether the differential equation

$$(x^2+t^2)\left(\frac{dx}{dt}\right)+2tx=e^t$$

is exact, and, if so, find a general solution.

Answer

Consider

$$(x^2+t^2)\left(\frac{dx}{dt}\right)+2tx-e^t=0$$

General case $p(x,t)\left(\frac{dx}{dt}\right)+q(x,t)=0$

Condition is $\frac{\partial p}{\partial t}(x,t)=\frac{\partial q}{\partial x}(x,t)$ for exact solution.

$p(x,t)=x^2+t^2$; $\frac{\partial p}{\partial t}=2t$

$q(x,t)=2tx-e^t$; $\frac{\partial q}{\partial x}=2t=\frac{\partial p}{\partial t}$ Equation is exact.

Now

$$\begin{aligned} p(x,t) &= x^2+t^2=\frac{\partial h}{\partial x} && (1)\\ q(x,t) &= 2tx-e^t=\frac{\partial h}{\partial t} && (2)\end{aligned}$$

Full equation is $\frac{\partial h}{\partial t}+\frac{\partial h}{\partial x}\frac{dx}{dt}+\frac{dh}{dt}=0$

Solution will be $h(x,t)=\text{constant}$

Solve(1) and (2) simultaneously to obtain $h(x,t)$:

Integrate (1):

$$h(x,t)=\frac{1}{3}x^3+xt^2+f(t) \tag{3}$$

Integrate (2):

$$h(x,t)=xt^2-e^t+g(x) \tag{4}$$

Reconcile (3) and (4)

$$h(x,t)=\frac{1}{3}x^3+xt^2-e^t$$

[identify $g(x)=\frac{1}{3}x^3$ and $f(t)=-e^t$]

Final solution

$$\frac{1}{3}x^3+xt^2-e^t=\text{constant}$$