

Question

Find the general solution of the equation

$$t \frac{dx}{dt} = x + te^{\frac{x}{t}}$$

Answer

$$t \frac{dx}{dt} = x + te^{\frac{x}{t}} \Rightarrow \frac{dx}{dt} = \frac{x}{t} + e^{\frac{x}{t}}$$

This is of the form $\frac{dx}{dt} = f\left(\frac{x}{t}\right)$ So let $y = \frac{x}{t} \Rightarrow \frac{dx}{dt} = t \frac{dy}{dt} + y$
So we can rewrite as

$$t \frac{dy}{dt} + y = y + e^y$$

$$t \frac{dy}{dt} = e^y$$

$$\int e^{-y} dy = \int \frac{dt}{t}$$

$$\Rightarrow -e^{-y} = \ln |t| + \text{constant}$$

$$t = Ae^{-(e^{-y})}$$