Question

Evaluate the triple integral

$$\iiint \sqrt{x^2+y^2}\, d(x,y,z)$$

where S is the solid sphere $x^2+y^2+z^2 \leq 1$.

Answer

Transform the spherical polar coordinates $(r,0,\phi)$ where $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$ and the sphere is described by the inequalities:

$$\left.\begin{array}{rcccl} 0 & \leq & r & \leq & 1 \\ 0 & \leq & \theta & \leq & \pi \\ 0 & \leq & \phi & \leq & 2\pi \end{array}\right\} \sqrt{x^2+y^2} = \sqrt{r^2\sin^2\theta(\cos^2\theta+\sin^2\phi)} = r\sin\theta$$

and $d(x,y,z) = r^2\sin\theta\, dr\, d\theta\, d\phi$.

$$\begin{aligned} I &= \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1}(r\sin\theta)(r^2\sin\theta)\, dr\, d\theta\, d\phi \\ &\quad \text{and since the limits are indenpendent of r}, \theta \text{ and } \phi: \\ &= \int_0^{2\pi} d\phi \int_0^1 r^3 dr \int_0^{\pi} \sin^2\theta\, d\theta \\ &= [\phi]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 \int_0^{\pi}\frac{1}{2}(1-cos2\theta)\, d\theta \quad [\text{using } 2\sin^2\theta = 1-\cos 2\theta] \\ &= \frac{\pi}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{\pi}{4}(\pi-0) = \frac{\pi^2}{4}. \end{aligned}$$