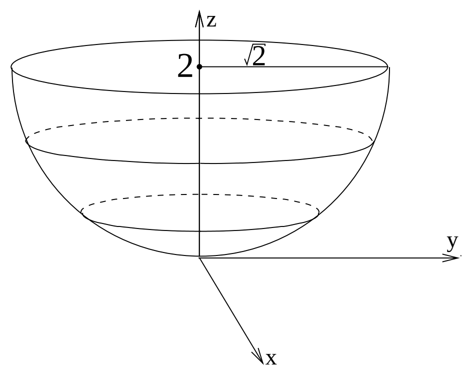
Question Sketch the region defined by the inequalities $x^2 + y^2 \leq z$, $0 \leq z \leq 2$. If the region is occupied by a solid whose density at the point (x, y, z) is $(3 - z)$, calculate its total mass by means of a triple integral. (HINT: Transform to cylindrical co-ordinates.)

Answer

For each z, $x^2+y^2 \leq z$ is a disc of radius $\sqrt{z}$. If we let z vary as $0 \leq z \leq 2$ we obtain the region:



Volume of region $= \iiint (3 - z) d(x, y, z)$

We use the cylindrical coordinates $(\rho, \phi\ z)$ where $x = \rho \cos\phi$ and $y = \rho \sin\phi$. Since $x^2 + y^2 = \rho^2(\cos^2\phi + \sin^2\phi) = \rho^2$, the region is defined by the inequalities: $\rho^2 \leq z \Rightarrow 0 \leq \rho \leq \sqrt{z}, \quad 0 \leq z \leq 2$ with any ϕ so that $0 \leq \phi \leq 2\pi$, Using $d(x, y, z) = \rho\, d\phi\, d\rho\, dz$

$$\begin{aligned}
\text{Volume of region} &= \int_{z=0}^{z=2} \int_{\rho=0}^{\rho=\sqrt{z}} \int_{\phi=0}^{\phi=2\pi} (3 - z)\rho\, d\phi\, d\rho\, dz \\
&= \int_{z=0}^{z=2} \int_{\rho=0}^{\rho=\sqrt{z}} [(3 - z)\rho\phi]_{\phi=0}^{\phi=2\pi}\ d\rho\, dz \\
&= 2\pi \int_{z=0}^{z=2} \int_{\rho=0}^{\rho=\sqrt{z}} (3 - z)\rho\, d\rho\, dz
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi \int_{z=0}^{z=2} \left[\frac{1}{2}(3-z)\rho^2\right]_{\rho=0}^{\rho=\sqrt{z}} dz \\
&= \pi \int_0^2 (3-z)z dz \\
&= \pi \int_0^2 3z - z^2 dz \\
&= \pi \left[\frac{3}{2}z^2 - \frac{1}{3}z^3\right]_0^2 \\
&= \pi \left[6 - \frac{8}{3}\right] \\
&= \frac{10\pi}{3}
\end{aligned}
$$