Question

A cash-or-nothing call is an option which pays at expiry \$1 at expiry T if the spot price is above the strike K and nothing if $S \leq K$. A cash-or-nothing put is an option which pays out nothing if $S > K$ and \$1 if $S \leq K$. Let C_b and P_b denote the values of cash-or-nothing calls and puts respectively.
Assuming both options have the same expiry date T, derive the put-call parity relation

$$C_b + P_b = e^{-r(T-t)}.$$

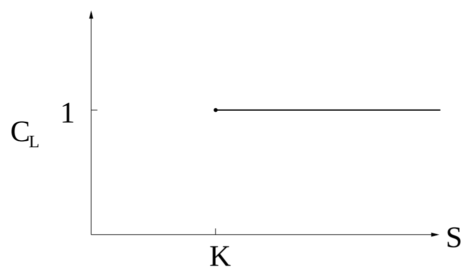
By considering relevant payoff diagrams, show that the payoff for C_b is equivalent to the delta of a vanilla European call option, with the same strike, at expiry. By differentiating the Black-Scholes equation with respect to S show that the value of a cash-or-nothing call option on an underlying which pays no dividend yield is equal to the delta of a vanilla European call option written on an underlying which pays a continuous dividend yield q^* and different interest rate r^* and determine q^* and r^*.
Hence show that C_b is given by

$$C_b(S,t) = N(d_2).$$

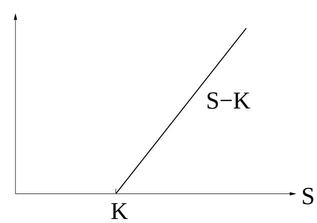
Answer
If we hold both C_b and P_b then we are guaranteed \$1 at expiry. Hence $C_b + P_b$ =present value of \$1, i.e.

$$C_b + P_b = e^{-r(T-t)}$$

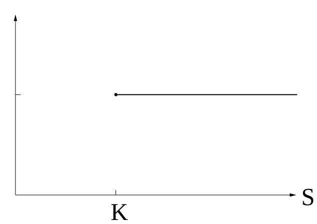
Payoff for C_b is



Payoff for normal call is



and $\Delta = \frac{\partial C}{\partial S}$ at expiry is



i.e. at expiry, $C_b = \Delta$.
Now C_b satisfies

$$\frac{\partial C_b}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_b}{\partial S^2} + rS\frac{\partial C_b}{\partial S} - rC_b = 0 \longleftarrow (1)$$

and call satisfies, say,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r^* - q^*)S\frac{\partial V}{\partial S} - r^* V = 0 \longleftarrow (2)$$

and at $t = T$, $C_b = \frac{\partial V}{\partial S} = \Delta$

$$\frac{\partial}{\partial S}(2) \Rightarrow \quad \frac{\partial \Delta}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Delta}{\partial S^2} + \sigma^2 S \frac{\partial \Delta}{\partial S}$$
$$+(r^* - q^*) S \frac{\partial \Delta}{\partial S} + (r^* - q^*)\Delta - r^* \Delta = 0$$

where $\Delta = \frac{\partial V}{\partial S}$, i.e.

$$\frac{\partial \Delta}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Delta}{\partial S^2} + (\sigma^2 + r^* - q^*) S \frac{\partial \Delta}{\partial S} - q^* \Delta \longleftarrow (3)$$

Can make (1) ≡ (3) by taking $q^* = r$, $\sigma^2 + r^* - q^* = r$

So

$$q^* - r, \quad r^* = r - \sigma^2 + q^* = 2r - \sigma^2.$$

Then then ensures that $C_b = \Delta$ since they are equal at expiry and satisfy the same PDE.

Δ for the regular call is (formula sheet)

$$\begin{aligned} \Delta = N(d_1^*) \quad d_1^* &= \frac{\log(S/E) + (r^* - q^* + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\log(S/E) + (2r - \sigma^2 - r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= d_2 \quad \text{for q = 0 case} \end{aligned}$$

$\Rightarrow C_b = N(d_2)e^{-r(T-t)}$.