Question

Assume that an asset S has growth rate μ, volatility σ and pays a continuous dividend yield q and that it evolves according to the stochastic differential equation

$$\frac{dS}{S} = (\mu - q)dt + \sigma dX$$

where dX is a Wiener process with the properties that

$$\begin{aligned} \mathcal{E}(dX) &= 0 \\ \mathcal{E}(dX^2) &= dt \\ \lim_{dt \to 0} dX^2 = dt \end{aligned}$$

Give a heuristic derivation of It$\underline{\text{o}}$'s lemma for a sufficiently differentiable function $V(S,t)$ which depends on both S and t.

Suppose that an option is written on this asset with the properties that at expiry it is equal to the asset, and prior to its expiry it pays out a known sum $K(S,t)dt$ during each time interval $(t, t+dt)$. By constructing an instantaneously risk-free portfolio and considering cash flows, show that it value V must satisfy the problem

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = -K(S,t)$$

$$t < T, \qquad V(S,T) = S.$$

Show that if $K(S,t)$ has the form $g(t)S$ where $g(t)$ is a known function of time, then there are solutions of the form $V = f(t)S$. Assuming that V does have this form find $V(S,t)$. Hence show that the delta for such an option is

$$\Delta(S,t) = e^{-q(T-t)} + \int_t^T e^{-q(s-t)} g(s)\, ds.$$

Answer
Itô asserts that if $f = f(S,t)$ then

$$\begin{aligned} df &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + O(dt) \quad \text{(Taylor series!)} \\ &= \frac{\partial v}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt + O(dt) \end{aligned}$$

Since $dS^2 = S^2((\mu - q)dt + \sigma dX)^2 = S^2 dX^2 + \cdots = S^2 dt$
Set up portfolio $\Pi = V - \Delta S$ where Δ is previsible, (i.e. $d(\Delta S) = \Delta dS$), i.e. Δ is fixed during time step dt. Then

$$\begin{aligned} d\Pi &= dV - \Delta dS + K(S,t)dt(\leftarrow \text{cash flow from option}) \\ &\quad -\Delta q S dt(\leftarrow \text{cash flow from dividend}) \\ &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta dS + K dt - \Delta q S dt \end{aligned}$$

Make Π risk free by putting $\Delta = \dfrac{\partial V}{\partial D}$, so all dX terms are eliminated;

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + K - \Delta q S\right) dt \\ &= r\Pi dt \end{aligned}$$

(interest earned on Π, since Π is riskfree and must grow at risk free rate.)
Thus

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + K - \Delta q S &= r(V - \Delta S) \\ \Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{dS^2} + (r-q)S\frac{\partial V}{\partial S} - rV &= -K \quad \text{as } \Delta = \frac{\partial \mathrm{V}}{\partial \mathrm{S}} \end{aligned}$$

If $K = g(t)S$ then we have

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r-q)SV_S - rV = -g(t)S$$

so if we try $V = f(t)S$ we get

$$\dot{f}(t)S + (r-q)f(t)S - rf(t)S = -g(t)S$$

which reduces to the ODE

$$\dot{f}(t) - qf(t) = -g(t)$$

So the form $V = f(t)S$ is consistent. From $V(S,T) = S$, we see that $f(T) = 1$. Thus we have to solve

$$\dot{f} - qf = -g \quad f(T) = 1$$

i.e

$$\begin{aligned}
\frac{d}{dt}(e^{-qt} f) &= -g e^{-qt}, \quad f(T) = 1 \\
\Rightarrow \int_t^T \frac{d}{dS}(e^{-qs} f(s)\, ds &= e^{-qT} f(T) - e^{-qt} f(t) = \int_t^T e^{-qs} g(s)\, ds \\
\Rightarrow f(t) &= e^{-q(T-t)} + \int_t^T e^{-q(s-t)} g(s)\, ds
\end{aligned}$$

Obviously if $V = f(t)S$, $\Delta = \frac{\partial V}{\partial S} = f(t)$, and

$$V = \left(e^{-q(T-t)} + \int_t^T e^{-q(s-t)} g(s)\, ds \right) S.$$