Question

Let $\mathcal{O}$ be the opportunity set, in risk/return space, for a set of risky assets (none of which are perfectly negatively correlated). Assume that short selling is allowed and that there is also a riskless investment with return R_F available. Suppose that an investor choses to invest $X \geq 0$ in a purely risky portfolio $\mathcal{P}$, with variance $\sigma_{\mathcal{P}}^2$ and expected return $R_{\mathcal{P}}$, and $1-X$ in the riskless investment. Show that as X varies the investor's total portfolio lies along a line of slope $(R_{\mathcal{P}} - R_F)/\sigma_{\mathcal{P}}$ and intercept R_F. Hence or otherwise, deduce that the problem of finding the capital market line reduces to maximizing the slope of the line above all possible portfolios in $\mathcal{O}$.

Consider a situation where there are three risky assets S_1, S_2 and S_3 with respective expected returns

$$R_1 = 0.1, \quad R_2 = 0.12, \quad R_3 = 0.18,$$

variances and covariances given by

$$\sigma_1^2 = 0.0016, \quad \sigma_{12} = 0.0016, \quad \sigma_{13} = 0,$$

$$\sigma_2^2 = 0.01, \quad \sigma_{23} = 0.0012, \quad \sigma_3^2 = 0.0144.$$

Further, assume that the risk free rate is 0.06, short selling and borrowing are allowed. Show that under these circumstances, the market price of risk is

$$\theta = \frac{167}{\sqrt{13861}} \sim 1.418$$

and that the optimal portfolio of risky assets consists of the following proportions of total wealth invested in S_1, S_2 and S_3, respectively,

$$\begin{array}{cc} \frac{237}{331} & \sim \ 0.716 \\ \frac{12}{331} & \sim \ 0.036 \\ \frac{82}{331} & \sim \ 0.247 \end{array}$$

Answer

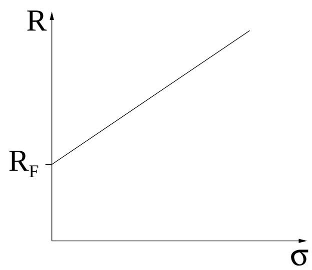
Let $\Pi = XP + (1-X)R_f$, R_f = risk free investment.

Then

$$\begin{aligned} R_\Pi = \text{expected return on } \Pi &= E(XP + (1-X)R_f) \\ &= XE(P) + (1-X)E(R_f) \\ &= XR_P + (1-X)R_F \end{aligned}$$

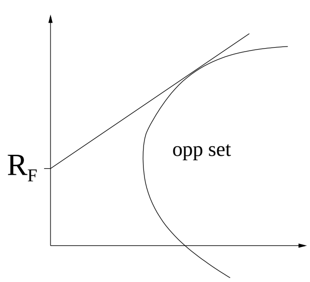
$$\begin{aligned} \sigma_\Pi^2 = \text{variance of } \Pi &= E((XP+(1-X)R_f)^2) \\ &\quad -E(XP+(1-X)R_f)^2 \\ &= E(X^2P^2 + 2X(1-X)PR_f + (1-X)^2R_f^2) \\ &\quad -X^2R_P^2 - 2X(1-X)R_PR_F - (1-X)^2R_F^2 \\ &= X^2(E(P^2)) - R_P^2) + 2X(1-X)[R_FE(P) \\ &\quad -R_FR_P] + (1-X)^2(E(R_f^2) - R_F^2) \\ &= X^2\mathrm{Var(P)} = \mathrm{X}^2\sigma_\mathrm{P}^2 \quad (\text{note X} \geq 0) \end{aligned}$$

$\Rightarrow \sigma_\Pi$, $R_\Pi = XR_P + (1-X)R_F$, so



$$\begin{aligned} \Rightarrow R_\Pi &= \tfrac{\sigma_\Pi}{\sigma_P} + \left(1 - \tfrac{\sigma_\Pi}{\sigma_P}\right) R_F \\ &= R_F + \left(\frac{R_P - R_F}{\sigma_P}\right)\sigma_\Pi \end{aligned}$$

Capital market line is just the straight line through the risk free rate R_F at $\sigma = 0$ which is tangent to opportunity sets boundary; (can assume opp set convex!)



Evidently it is the line passing through $(\mathrm{R_F}, 0)$ and some risky portfolio which has greatest possible slope.

Aim is to maximize $\dfrac{R_P - R_F}{\sigma_P}$ over all possible risky assets.

$$\begin{aligned} LetP &= X_1S_1 + X_2S_2 + X_3S_3 \quad \text{with } \mathrm{X_1 + X_2 + X_3 = 1} \\ R+P &= X_1R_1 + X_2R_2 + X_3R_3 \\ \sigma_P &= (X_1^2\sigma_1^2 + 2X_1X_2\sigma_{12} + X_2^2\sigma_2^2 + 2X_2X_3\sigma_{23} + X_3^2\sigma_3^2 \\ &\quad +2X_1X_3\sigma_1\sigma_3)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \text{Thus} \quad \mathrm{R_P - R_F} &= (4X_1 + 6X_2 + 12X_3)/100 \\ \sigma + P &= (16X_1^2 + 32X_1X_2 + 100X_2^2 + 24X_2X_3 \\ &\quad +144X_3^2)^{\frac{1}{2}}/100 \end{aligned}$$

So we want to maximize

$$(*) \quad \theta = \frac{4X_1 + 6X_2 + 12X_3}{(16X_1^2 + 32X_1X_2 + 100X_2^2 + 24X_2X_3 + 144X_3^2)^{\frac{1}{2}}}$$

$$= \frac{4X_1 + 6X_2 + 12X_3}{\alpha}$$

$$\begin{aligned} \frac{\partial\theta}{\partial X_1} \quad & \frac{4}{\alpha} - \frac{1}{2}\frac{(4X_1 + 6X_2 + 12X_3)}{\alpha^3}(32X_1 + 32X_2) = 0 \\ & \Rightarrow \frac{4X_1 + 6X_2 + 12X_3}{\alpha^2}(32X_1 + 32X_2) = 8 \\ \frac{\partial\theta}{\partial X_2} \quad & = 0 \\ & \Rightarrow \frac{4X_1 + 6X_2 + 12X_3}{\alpha^2}(32X_1 + 200X_2 + 24X_3) = 12 \\ \frac{\partial\theta}{\partial X_3} \quad & = 0 \\ & \Rightarrow \frac{4X_1 + 6X_2 + 12X_3}{\alpha^2}(24X_2 + 288X_3) = 24 \end{aligned}$$

Put $z_i = \left(\frac{4X_1 + 6X_2 + 12X - 3}{\alpha^2}\right) X_i$ so we get

$$\begin{pmatrix} 32 & 32 & 0 \\ 32 & 200 & 24 \\ 0 & 24 & 288 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \\ 24 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 & 4 & 0 \\ 16 & 100 & 12 \\ 0 & 1 & 12 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 4 & 4 & 0 & 1 \\ 16 & 100 & 12 & 6 \\ 0 & 1 & 12 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & 4 & 0 & 1 \\ 0 & 84 & 12 & 2 \\ 0 & 1 & 12 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & 4 & 0 & 1 \\ 0 & 83 & 0 & 1 \\ 0 & 1 & 12 & 1 \end{array}\right)$$

So

$$\begin{aligned} z_2 &= 1/83 \\ 4z_1 &= 1 - 4/83 = 79/83 \\ 12z_3 &= 1 - 1/83 = 82/83 \end{aligned}$$

$\Rightarrow \quad z_1 = 79/(4 \times 83),\ z_2 = 1/83,\ 82/(12 \times 83)$
Now note that

$$\begin{aligned} z_1 + z_2 + z_3 &= \left(\frac{4X_1 + 6x_2 + 12X_3}{\alpha^2}\right)(X_1 + X_2 + X_3) \\ &= \left(\frac{4X_1 + 6X_2 + 12X_3}{\alpha^2}\right) \end{aligned}$$

hence $X_i = z_i/(z_1+z_2+z_3)$; $z_1+z_2+z_3 = \frac{1}{12\times 83}(82+3\times 79+12) = \frac{1}{12\times 83}(331)$

$$\begin{aligned} i.e. \quad X_1 &= \tfrac{79}{4\times 83} \times \tfrac{12\times 83}{331} = \frac{237}{331} \\ X_2 &= \tfrac{1}{83} \times \tfrac{12\times 83}{331} = \frac{12}{331} \\ X_3 &= \tfrac{82}{12\times 83} \times \tfrac{12\times 83}{331} = \frac{82}{331} \end{aligned}$$

Finally put these numbers back into $(*)$ to give

$$\theta = \frac{167}{\sqrt{13861}}$$

$$\begin{aligned} z_1 &= \frac{79}{332} = \frac{237}{996} \\ z_2 &= \frac{1}{83} = \frac{12}{996} \\ z_3 &= \frac{82}{996} \end{aligned}$$