Question

Find the general solution to each of the following equations

1. $x^2y'' - 2y = 0$
2. $x^2y'' + xy' + 4y = 0 \qquad (*)$
3. $x^2y'' - 3xy' + 4y = \ln x \qquad (*)$
4. $x^2y'' + 7xy' + 5y = x \qquad (*)$

Answer

For all the following problems note that if $t = \ln x, \quad x = e^t$ and $y(x) = Y(t)$ then $\dfrac{dy}{dx} = e^{-t}\dfrac{dY}{dt}, \quad \dfrac{d^2y}{dx^2} = e^{-2t}\dfrac{d^2Y}{dt^2} - e^{-2t}\dfrac{dY}{dt}$.

a) $x^2y'' - 2y = 0 \Rightarrow e^{2t}\left(e^{-2t}Y'' - e^{-2t}Y'\right) - 2Y = 0$

$Y'' - Y' - 2Y = 0$, try $Y = e^{mt}, \quad m^2 - m - 2 = 0, \quad m = -1, \quad m = 2$

$Y = Ae^{-t} + Be^{2t} \Rightarrow y = A\dfrac{1}{x} + Bx^2$

b) $x^2y'' + xy' + 4y = 0 \Rightarrow e^{2t}\left(e^{-2t}Y'' - e^{-2t}Y'\right) + e^t\left(e^{-t}Y'\right) + 4Y = 0$

$Y'' + +4Y = 0$, try $Y = e^{mt}, \quad m^2 + 4 = 0, \quad m = 2i, \quad m = -2i$

$Y = Ae^{2it} + B^{-2it}$ or (more sensibly), $Y = C\cos(2t) + D\sin(2t)$

$y = C\cos(2\ln x) + D\sin(2\ln x)$

c) $x^2y'' - 3xy' + 4y = \ln x$

$\Rightarrow e^{2t}\left(e^{-2t}Y'' - e^{-2t}Y'\right) - 3e^t\left(e^{-t}Y'\right) + 4Y = \ln\left(e^t\right)$

$\Rightarrow Y'' - 4Y' + 4Y = t$

First find $Y_c, \quad Y'' - 4Y + 4Y = 0 \Rightarrow m^2 - 4m + 4 = 0$

$\Rightarrow m = 2$ (repeated). So $Y_c = Ae^{2x} + Bxe^{2x}$

Now find Y_{PI} by either variation of parameters or undetermined constants.

Try $Y_{PI} = Dt + E \Rightarrow 0 - 4(D) + 4(Dt + E) = t \Rightarrow 4D = 1$

$$\Rightarrow D = \frac{1}{4}, \quad -4D + 4E = 0 \Rightarrow E = \frac{1}{4}$$

$$Y_{PI} = \frac{1}{4}t + \frac{1}{4}$$

$$Y = Ae^{2x} + Bxe^{2x} + \frac{1}{4}t + \frac{1}{4} \Rightarrow y(x) = Ax^2 + Bx^3 + \frac{1}{4}\ln t + \frac{1}{4}$$

d) $x^2y'' + 7xy' + 5y = x$

$$\Rightarrow e^{2t}\left(e^{-2t}Y'' - e^{-2t}Y'\right) + 7e^t\left(e^{-t}Y'\right) + 5Y = e^t$$

$$\Rightarrow Y'' + 6Y' + 5Y = e^t$$

First find Y_c, $\quad Y_c'' + 6Y_c' + 5Y_c = 0$

$$\Rightarrow m^2 - 6m + 5 = 0, \quad m = -1, -5, \quad Y_c = Ae^{-t} + Be^{-5t}$$

Find Y_{PI} by variation of parameters or undetermined coefficients.

Try $Y_{PI} = De^t \Rightarrow (D + 6D + 5D)e^t = e^t \Rightarrow D = \frac{1}{12}$

$$Y_{PI} = \frac{1}{12}e^t$$

$$Y = Ae^{-t} + Be^{-5t} + \frac{1}{12}e^t \Rightarrow y(x) = A\left(\frac{1}{x}\right) + B\left(\frac{1}{x}\right)^5 + \frac{1}{12}x$$