QUESTION

(a) Define the direct product of two groups $(G, e_g, *)$ and $H, e_H, .)$ and prove that the direct product of $(G, e_g, *)$ and $(H, e_H, .)$ is isomorphic to the direct product of $(H, e_H.)$ and $(G, e_g, *)$

(b) State the internal direct products theorem. Use it to prove that:

- **(i)** the group of symmetries of a regular hexagon is isomorphic to $S_3 \times Z_2$ (you may assume the classification of groups of order 6).
- **(ii)** The permutations (123) and (45) generate a subgroup of S_5 isomorphic to the cyclic group Z_6.

ANSWER

(a) The direct product has $G \times H = \{(g, h)|g \in G, h \in H\}$ for its elements, (e_G, e_H) for its identity element and group operation $(g_1, h_1)@(g_2, h_2) = (g_1 * g_2, h_1.h_2)$. The function $\phi((g, h)) = (h, g)$ defines a map $G \times H \longleftarrow H \times G$ which is bijective. We will denote multiplication in $H \times G$ by juxtaposition.

(b) The internal direct product theorem states that if H and K are subgroups of a group G such that $H \cap K = \{e\}$ and every element $h \in H$ commutes with every element $k \in K$ then the subgroup they generate, $\langle H \cup K\rangle$, is isomorphic to the direct product $H \times K$.

- **(i)** The subgroup H generated by the rotation of π around the center is central of order 2. The subgroup K generated by reflections in lines joining opposite vertices is a non-abelian group of order 6 so is isomorphic to S_3. Since the angle between any two of these lines is $\frac{2\pi}{3}$ this subgroup does not contain the rotation of π so $H \cap K = \{e\}$, and by centrality of H, $hk = kh$ for any $h \in H$, and any $k \in K$, So $\langle h \cup K\rangle$ is isomorphic to $S_3 \times z_2$ as required.
- **(ii)** The cycles (123) and (45) are disjoint and commute so they generate commuting cyclic subgroups H and K of orders 2,3 respectively. Clearly $H \cap K = \{e\}$ so they generate a group isomorphic to $Z_2 \times Z_3$ which is isomrphic to Z_6 since hcf(2,3)=1.