QUESTION

(a) Define the terms *homomorphism* and *isomorphism*.

(b) In each of the following cases decides whether or not the given pair of groups are isomorphic. If they are give an explicit isomorphism between them, and if they are not give brief reasons how you can be sure of this.

(i) $Z_3 \times Z_5$ and Z_{15}.

(ii) $Z_3 \times Z_6$ and Z_{18}.

(iii) D_5 and Z_{10}.

(c) For each positive integer n less than or equal to 8 give a list of groups of order n so that no two groups in your list are isomorphic, but every group of order n is isomorphic to one of the groups on your list. For each order where there is more than one isomorphism class of group of that order, indicate how you distinguish between the different isomorphism classes.

(d) Decide which of the groups in your list from part (iii) is isomorphic to the group given by the following Cayley table, giving brief reasons for your answer:

.	e	g	c	b	h	d	f	k
e	e	g	c	b	h	d	f	k
g	g	c	b	e	k	h	d	f
c	c	b	e	g	f	k	h	d
b	b	e	g	c	d	f	k	h
h	h	d	f	k	c	b	e	g
d	d	f	k	h	g	c	b	e
f	f	k	h	d	e	g	c	b
k	k	h	d	f	b	e	g	c

ANSWER

(a) Given groups $(G, e_G, *)$ and $(H, e_H, .)$ a homomorphism is a function $\phi : G \Longrightarrow H$ such that for every $g, k \in G, \phi(g * h) = \phi(g), \phi(h)$. An isomorphism is a bijective homomorphism.

(b) **(i)** The function $Z_3 \times Z_3 \Longrightarrow Z_{15}$ defined by $(1, 1) \mapsto 1$ is an isomorphism since hcf(3,5)=1.

(ii) $Z_3 \times z_6$ is not isomorphic to Z_{18} since every element of $Z_3 \times Z_6$ has order dividing lcm(3,6)=6.

(iii) D_5 is not abelian so it cannot be isomorphic to Z_{10}.

(c)

Group	order	distinguishing features
$\{e\}$	1	
Z_2	2	
Z_3	3	
Z_4	4	contains an element of order 4, and two elements of order 2
$z_2 \times Z_2$	2	3 elements of order 2
$Z)_5$	5	
Z_6	6	abelian
D_3	6	non-abelian
Z_7	7	
Z_8	8	abelian with one element of order 2
$Z_4 \times Z_2$	8	abelian with three elements of order 2
$Z_2 \times Z_2 \times Z_2$	8	abelian with seven elements of order 2
D_4	8	non-abelian with five elements of order 2
$\mathcal{Q}$	8	non-abelian with one element of order 2

(d) $hg = d \neq k = gh$ so the group is non-abelian. It has one element of order 2 so it is isomorphic to $\mathcal{Q}$.