Exam Question
Topic: Double Integral
Let R denote the region in the first quadrant $(x > 0, y > 0)$ bounded by the hyperbolas $xy = 1, xy = 9$ and the lines $y = x, y = 4x$.
Use the change of variable given by $x = u/v, y = uv$ to evaluate the double integral

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right)\, d(x, y)$$

Solution

$$x = \frac{u}{v}, y = uv \quad \text{so} \quad xy = u^2, \frac{y}{x} = v^2$$

The Jacobian for this transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$$

$$\text{So} \iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right)\, d(x, y) = \iint_{R'} (v + u).\frac{2u}{v}\, d(u, v)$$

$$= \int_{u=1}^{3} du \int_{v=1}^{2} \left(2u + \frac{2u^2}{v}\right) dv = \int_{u=1}^{3} \left[2uv + 2u^2 \ln v\right]_{v=1}^{2} du$$

$$= \int_1^3 (2u + 2u^2 \ln 2)\, du = \left[u^2 + \frac{2}{3}u^3 \ln 2\right]_1^3$$

$$= 8 + \frac{2}{3} \times 26 \times \ln 2 = 20.0146 \text{ (4 d.p.)}$$