QUESTION

Write down the formula for the value of $\phi(n)$ in terms of the prime factorisation of n and hence find

(i) all n for which $\phi(n) = \frac{4n}{11}$.

(ii) all n for which $\phi(n) = 2$.

(iii) all n for which $\phi(n) = 12$.

ANSWER

$\phi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)$.

(i) If $\phi(n) = \frac{4n}{11}$, then $\prod_{p|n}\left(1 - \frac{1}{p}\right) = \frac{4}{11}$, i.e. $\prod_{p|n}\left(\frac{p-1}{p}\right) = \frac{4}{11}$.

In the expression $\prod_{p|n}\left(\frac{p-1}{p}\right)$, the largest prime occurring in n appears in the denominator, but cannot cancel with anything in the numerator. Thus it re3mains on the denominator in the simplified expression for $\prod_{p|n}\left(\frac{p-1}{p}\right)$, so we may deduce that the largest prime appearing in n is 11. Then $\prod_{p|n}\left(\frac{p-1}{p}\right) = \prod_{p|n,p<11}\left(\frac{p-1}{p}\right).\frac{10}{11} = \frac{4}{11}$. Thus $\prod_{p|n,p<11}\left(\frac{p-1}{p}\right) = \frac{2}{5}$, and so we can similarly deduce that the next largest prime occurring is 5. Thus $\prod_{p|n,p<5}\left(\frac{p-1}{p}\right).\frac{4}{5} = \frac{2}{5}$, so $\prod_{p|n,p<5}\left(\frac{p-1}{p}\right) = \frac{1}{2}$ and we may now deduce that the only prime occurring is 2. Thus $\phi(n) = \frac{4n}{11}$ if and only if $n = 2^{\alpha}.5^{\beta}.11^{\gamma}$ for $\alpha,\ \beta,\ \gamma$ all positive integers.

(ii) $\phi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right) = n\prod_{p|n}\left(\frac{p-1}{p}\right)$. Thus if $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$, then $\phi(n) = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\ldots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\ldots(p_k-1)$. Now suppose $\phi(n) = 2$. If p_i is a prime > 2, then $p_i \not|\phi(n)$, so $\alpha_i = 1$. Moreover, $(p_i - 1)$ divides $\phi(n)$ $(= 2)$, so p_i can only be 3. Thus n takes one of the forms 2^{α_1} or $2^{\alpha_1}.3$ or 3. To see which integers of these forms are allowed, note

$\phi(2^{\alpha_1}) = 2^{\alpha_1}\left(1 - \frac{1}{2}\right) = 2^{\alpha_1-1} = 2$ only if $\alpha_1 = 2$

$\phi(2^{\alpha_1}.3) = \phi(2^{alpha_1})\phi(3) = 2^{\alpha_1-1}.3\left(1 - \frac{1}{3}\right) = 2^{\alpha_1} = 2$ only if $\alpha_1 = 1$

$\phi(3) = 3\left(1 - \frac{1}{3}\right) = 2$

Thus the possible cases are 2,4 and 6.

(iii) If $\phi(n) = 12$, we may argue in the same way as above, and deduce that if $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$, then $\alpha_i = 1$ for all primes occurring except those which divide 12, namely 2 and 3. Moreover, if p_i occurs, then $(p_i - 1)$ divides 12, so p_i can only be 13, 7, 5, 3 or 2.

Since $\prod_{p|n}(p-1)$ must divide 12, we see that the only prime that can appear together with 13 is 2, so the possibilities involving 13 are 13 and $2^\alpha.13$, and a quick check shows that of these, only 13 and 26 satisfy $\phi(n) = 12$.

Similarly, the only primes that can occur with 7 are 3 and 2, so we must check 7, $3^\alpha.7, 2^\alpha.7$ and $2^\alpha.3^\beta.7$. We note $\phi(7) = 6 \neq 12, \phi(3^\alpha.7) = \phi(3^\alpha).\phi(7) = 3^{\alpha-1}.2.6$, so the only possibility is $\alpha = 1$, giving 21 as a possibility. $\phi(2^\alpha.7) = \phi(2^\alpha)\phi(7) = 2^{\alpha-1}.6 = 12$ only if $\alpha = 2$, so 28 is a possibility, and $\phi(2^\alpha.3^\beta.7) = \phi(2^\alpha)\phi(3^\beta).\phi(7) = 2^{\alpha-1}.3^{\beta-1}.2.6$, showing that $\beta = \alpha = 1$, so that 42 is a possibility. Thus the possibilities where $7|n$ are 21,28 and 42.

The case where $5|n$ is quickly eliminated:-

If $n = 5m$, where $\gcd(5, m) = 1$, then $\phi(n) = \phi(5)\phi(m) = 4\phi(m)$, so that $\phi(m) = 3$, contradicting cor.5.5 which tells us that $\phi(m)$ is always even if $m > 2$.

We are left with the possibilities $n = 2^\alpha,\ n = 3^\alpha$ or $n = 2^\alpha.3^\beta$, and again we may check the formulae to see that the only case giving $\phi(n) = 12$ is $n = 2^2.3^2 = 36$. Thus the full list of possibilities is 13, 21, 26, 28, 36 and 42.