

QUESTION

For what values of  $n$  does the equation  $\phi(2n) = \phi(n)$  hold?

ANSWER

If  $\gcd(2, n) = 1$  (i.e.  $n$  is odd), then as  $\phi$  is multiplicative,  $\phi(2n) = \phi(2)\phi(n) = (2 - 1)\phi(n) = \phi(n)$ . Thus the equation holds for all odd  $n$ .

If  $n$  is even,  $n = 2^r m$  say, where  $\gcd(2, m) = 1$ , then  $\phi(n) = \phi(2^r)\phi(m) = 2^r \left(1 - \frac{1}{2}\right)\phi(m) = 2^{r-1}\phi(m)$ , while  $\phi(2n) = \phi(2^{r+1}m) = \phi(2^{r+1})\phi(m) = 2^r\phi(m)$ . Thus  $\phi(n) \neq \phi(2n)$  if  $n$  is even. Hence  $\phi(n) = \phi(2n)$  if and only if  $n$  is odd.