Question

By explicitly expanding the following two determinants, prove the rule that the sign of a 3×3 determinant is changed by exchanging two rows:

$$det\ \mathbf{M_1} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad det\ \mathbf{M_2} = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Answer

$$\begin{aligned} det(\mathbf{m_1}) &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ & \text{Using minors, expanding by the 1st row} \\ & \text{and remembering the } + - + \text{ sign pattern.} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad (1) \end{aligned}$$

$$\begin{aligned} det(\mathbf{m_2}) &= \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ & \text{Using minors, expanding by the 1st row} \\ & \text{and remembering the } + - + \text{ sign pattern.} \\ &= a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\ &= a_2(b_1c_3 - b_3c_1) - b_2(a_1c_3 - a_3c_1) + c_2(a_1b_3 - a_3b_1) \quad (2) \end{aligned}$$

Now compare (1) and (2)
Expand (1)
$det\mathbf{m_1} = (1) = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1$
$det\mathbf{m_2} = (2) = -a_1b_2c_3 + a_1b_3c_2 + a_2b_1c_3 - a_3b_1c_2 - a_2b_3c_1 + a_3b_2c_1$
i.e., $(1) = -(2)$ or $det\mathbf{m_1} = det\mathbf{m_2}$.
Since all the $a_i's$, b_j' $c_k's$ are arbitrary. This rule holds for all 3×3 determinants. Hence if you exchange two rows the sign changes.