Vector Fields
Conservative Fields

Question
The function $\underline{F}$ is given by $\underline{F} = r^2 \cos\theta \underline{\hat{r}} + \alpha r^\beta \sin\theta \underline{\hat{\theta}}$. For what values of the constants α and β is $\underline{F}$ conservative? For these values find a corresponding potential.

Answer
As $\underline{F} = r^2 \cos\theta \underline{\hat{r}} + \alpha r^\beta \sin\theta \underline{\hat{\theta}} = \nabla\phi(r,\theta)$ we must have

$$\frac{\partial\phi}{\partial r} = r^2\cos\theta, \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \alpha r^\beta \sin\theta.$$

$$\begin{aligned} \Rightarrow \phi(r,\theta) &= \frac{r^3}{3}\cos\theta + C(\theta) \\ \text{and } C'(\theta) - \frac{r^3}{3} &= \frac{\partial\phi}{\partial\theta} \\ &= \alpha r^{\beta+1}\sin\theta. \end{aligned}$$

This can be solved for a function $C(\theta)$ which is independent of r if $\alpha = -1/3$ and $\beta = 2$.
In this case, $C(\theta) = C$, with C being a constant. $\underline{F}$ is conservative is the two constants α and β have the above stated values. A potential for $\underline{F}$ is $\phi = \frac{1}{3}r^3\cos\theta + C$.