Vector Fields
Conservative Fields

Question

(a) Show that the gradient of a function which is expressed in polar coordinates in the plane is

$$\nabla\phi(t,\theta) = \frac{\partial\phi}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{\hat{\theta}}.$$

(b) Use the result from part (a) to show that a necessary condition for the vector field

$$\underline{F}(r,\theta) = F_r(r,\theta)\underline{\hat{r}} + F_\theta(r,\theta)\underline{\hat{\theta}}$$

(expressed in polar coordinates) to be conservative is that

$$\frac{\partial F_r}{\partial\theta} - r\frac{\partial F_\theta}{\partial r} = F_\theta.$$

Answer

(a) As $x = r\cos\theta$ and $y = r\sin\theta$

$$\begin{aligned}
\frac{\partial\phi}{\partial r} &= \cos\theta\frac{\partial\phi}{\partial x} + \sin\theta\frac{\partial\phi}{\partial y} \\
\frac{\partial\phi}{\partial\theta} &= -r\sin\theta\frac{\partial\phi}{\partial x} + r\cos\theta\frac{\partial\phi}{\partial y}
\end{aligned}$$

And

$$\begin{aligned}
\underline{\hat{r}} &= \frac{x\underline{i} + y\underline{j}}{r} = (\cos\theta)\underline{i} + (\sin\theta)\underline{j} \\
\underline{\hat{\theta}} &= \frac{-y\underline{i} + x\underline{j}}{r} = -(\sin\theta)\underline{i} + (\cos\theta)\underline{j}
\end{aligned}$$

This leads to the fact that

$$\begin{aligned}
\frac{\partial\phi}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{\hat{\theta}} &= \left(\cos^2\theta\frac{\partial\phi}{\partial x} + \sin\theta\cos\theta\frac{\partial\phi}{\partial y}\right)\underline{i} \\
&\quad + \left(\cos\theta\sin\theta\frac{\partial\phi}{\partial x} + \sin^2\theta\frac{\partial\phi}{\partial y}\right)\underline{j} \\
&\quad + \left(\sin^2\theta\frac{\partial\phi}{\partial x} - \sin\theta\cos\theta\frac{\partial\phi}{\partial y}\right)\underline{i} \\
&\quad + \left(-\cos\theta\sin\theta\frac{\partial\phi}{\partial x} + \cos^2\theta\frac{\partial\phi}{\partial y}\right)\underline{j} \\
&= \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} = \nabla\phi.
\end{aligned}$$

(b) If $\underline{F}(r,\theta) = F_r(r,\theta)\underline{\hat{r}} + F_\theta(r,\theta)\underline{\hat{\theta}}$ is conservative, then $\underline{F} = \nabla\phi$ for a scalar field $\phi(r,\theta)$. Using part (a)

$$\frac{\partial\phi}{\partial r} = F_r, \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = F_\theta.$$

In order to have mixed second partial derivatives of ϕ that are equal, it is necessary to have

$$\frac{\partial F_r}{\partial\theta} = \frac{\partial}{\partial r}(rF_\theta) = F_\theta + r\frac{\partial F_\theta}{\partial r}$$

$$\text{i.e. } \frac{\partial F_r}{\partial\theta} - r\frac{\partial F_\theta}{\partial r} = F_\theta.$$