QUESTION

(b) The position of a particle P at a time t is given by

$$\mathbf{r} = \sin(2t)\mathbf{i} + \cos(2t)\mathbf{j} + t\mathbf{k}.$$

(i) Find the velocity and speed of P at time t.

(ii) Find the acceleration of P at time t, and deduce that the acceleration is perpendicular to the velocity.

(iii) Determine the angle between the particle's position $\mathbf{r}$ and the velocity at time t, and find the limiting value if this angle at large time.

ANSWER

(b) $\mathbf{r} = \sin(2t)\mathbf{i} + \cos(2t)\mathbf{j} + t\mathbf{k}$

(i) Velocity $= \dfrac{d\mathbf{r}}{dt} = 2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j} + \mathbf{k}$

Speed $= \left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{(2\cos(2t))^2 + (-2\sin(2t))^2 + 1^2}$

$= \sqrt{(4(\cos^2(2t) + \sin^2(2t)) + 1)} = \sqrt{5}$

(ii) Acceleration $= \dfrac{d^2\mathbf{r}}{dt^2} = -4\sin(2t)\mathbf{i} - 4\cos(2t)\mathbf{j}$

$\dfrac{d\mathbf{r}}{dt} \cdot \dfrac{d^2\mathbf{r}}{dt^2} = (2\cos(2t), -2\sin(2t), 1) \cdot (-4\sin(2t, -4\cos(2t), 0)$

$= -8\cos(2t)\sin(2t) + 8\sin(2t)\cos(2t) + 0 = 0$

Therefore the velocity is perpendicular to the acceleration.

(iii) $\cos(\theta) = \dfrac{\mathbf{r}.\frac{d\mathbf{r}}{dt}}{|\mathbf{r}||\frac{d\mathbf{r}}{dt}|}$

$= \dfrac{2\sin(2t)\cos(2t) - 2\cos(2t)\sin(2t) + t}{\sqrt{\sin^2(2t) + \cos^2(2t) + t^2}\sqrt{5}} = \dfrac{t}{\sqrt{5(1+t^2)}}$

Therefore $\theta = \cos^{-1}\left(\dfrac{t}{\sqrt{5(1+t^2)}}\right)$

As $t \to \infty,\ \theta \to \cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right) = 63.4^\circ$