Question
Explain what is meant by an irreducible Markov chain.
The transition probability matrices given below are for 4-state. 5-state and infinite Markov chains respectively, each with states labelled $1, 2, 3, \cdots$. In each case, determine if the Markov chain is irreducible and classify its states as positive-recurrent, null-recurrent or transient. Give the periods of any periodic states. For (ii) find the mean recurrence times of any ergodic states. State, but do not prove, any general results you use.

(i) $$\begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(ii) $$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}$$

(iii) $$\begin{pmatrix} p & 0 & 1-p & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ p & 0 & 0 & 1-p & 0 & 0 & 0 & \cdots \\ p & 0 & 0 & 0 & 1-p & 0 & 0 & \cdots \\ p & 0 & 0 & 0 & 0 & 1-p & 0 & \cdots \\ . & . & . & . & . & . & . & \cdots \end{pmatrix}, \; 0 < p < 1.$$

Answer
A Markov chain is a sequence of discrete (integer-valued) random variables (X_n) with the property that

$$P(X_{n+1} = i \mid X_0 = a_0,\ x_1 = a_1,\ \cdots\ X_n = j) = P(x_{n+1} = i \mid X_n = j)$$

A Markov chain is irreducible if all its states intercommunicate, i.e. if it is possible to pass between each pair of states in a finite number of steps with positive probability.

(i) Transition diagram

PICTURE

Irreducible, finite, so all states are positive recurrent. All have period 3.

(ii) Transition diagram

PICTURE

Not irreducible closed sets are $\{1,\ 3\}$ $\{2,\ 4\}$ all aperiodic and positive recurrent. State 5 is transient.

$\mu_1 = 1 \cdot \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \cdots = 2$

From symmetry $\mu_2 = \mu_3 \mu_4 = 2$

(iii) Transition diagram

PICTURE

Not irreducible (2) is absorbing $\{1,\ 3,\ 4,\ cdots\}$ is a closed set of states and so are all of the same type. (1) is aperiodic, so they all are

$$p_{11} = p + p(1-p) + p(1-p)^2 + \cdots = p\sum_{n=0}^{\infty}(1-p)^n = 1$$

so each state is recurrent

$$\mu_1 = p + 2p(1-p) + 3p(1-p)^2 + \cdots = \frac{1}{p} < \infty$$

so each state is positive recurrent.