

Multiple Integration
Iteration of Double Integrals

Question

Suppose $F'(x) = f(x)$ and $G'(x) = g(x)$ on the interval $a \leq x \leq b$.
 T is the triangle defined by the vertices (a, a) , (b, a) and (b, b) .
Iterate $\iint_T f(x)g(x) dA$ in both directions to show that

$$\begin{aligned} \int_a^b f(x)G(x) dx &= F(b)G(b) - F(a)G(a) \\ &\quad - \int_a^b g(y)F(y) dy \end{aligned}$$

Answer

$F'(x) = f(x)$ and $G'(x) = g(x)$ on $a \leq x \leq b$

$$\begin{aligned} \Rightarrow \iint_T f(x)g(x) dA &= \int_a^b f(x) dx \int_a^x G'(y) dy \\ &= \int_a^b f(x)(G(x) - G(a)) dx \\ &= \int_a^b f(x)G(x) dx - G(a)F(b) + G(a)F(a) \\ \iint_T f(x)g(x) dA &= \int_a^b g(y) dt \int_y^b F'(x) dx \\ &= \int_a^b g(y)(F(b) - F(y)) dy \\ &= F(b)G(b) - F(b)G(a) - \int_a^b F(y)g(y) dx \end{aligned}$$

$$\begin{aligned} \Rightarrow \Rightarrow \int_a^b f(x)G(x) dx &= F(b)G(b) - F(a)G(a) \\ &\quad - \int_a^b g(y)F(y) dy \end{aligned}$$