Question

By considering the function $A\log(z) + B$, find a harmonic function in the upper half plane $Im(z) > 0$ which takes the prescribed values

$$\phi(x, 0^+) = \begin{cases} 1, x > 0 \\ 0, x < 0 \end{cases}$$

Answer

Let $Phi = A\log z + B$, where $\Phi = Phi(z),\ z = x + iy$. Assume A and B are real.

Then

$Re(\Phi) = A\log(x^2 + y^2)^{\frac{1}{2}}$

$Im(\Phi) = A\arg(z) + B = \arctan\left(\frac{y}{x}\right) \times A + B$ fro $\underline{y > 0}$

Φ is analytic except at $z = 0$ (and along a cut from there which can be taken along negative imaginary axis). Thus $Re(\Phi)$ and $Im(\Phi)$ are harmonic in $Im(z) > 0$.

If $\phi(x, o^+) = \begin{cases} 1, x > 0 \\ 0, x < 0 \end{cases}$

the obvious choice is $Im(\Phi)$.

Why?

Well

$$\begin{aligned} Im(\Phi) &= A\theta + B \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

(1)

So for $\begin{matrix} x > 0,\ y = 0,\ \theta = 0 \\ x < 0,\ y = 0,\ \theta = \pi \end{matrix}$.

Thus

$Im(\Phi) = \begin{cases} B & x > 0,\ y = 0 \\ A\pi + B & x < 0,\ y = 0 \end{cases}$ (2)

Compare (1) and (2) to see $B = 1,\ A = \frac{-1}{\pi}$.

Thus harmonic function is:

$$\underline{\phi(x, y) = 1 - \frac{1}{\pi}\arctan\left(\frac{y}{x}\right)}$$

This is unique.